Recitation 03

To-do list:

1. An example of a set without the Archimedean property.
2. Discussing the intuition behind open sets with examples.

Let's get into it. The Archimedean property states that for all $x, y \in \mathbb{R}$ such that $0 < x < y$, there exists an $n \in \mathbb{N}$ such that $nx > y$. In other words, no real number is infinitely larger than any other real number. This property feels particularly trivial for the real numbers, but not all ordered sets have such a nice property. For instance, consider the set S of all polynomials with real valued coefficients. To write this formally, we state

$$S = \left\{ \sum_{i=0}^{n} a_i x^i \mid a_i \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

We can turn S into an ordered set by defining what it means for a polynomial to be "positive". For S, we will state that if $p(x) = \sum_{i=0}^{n} a_i x^i$ has $a_n > 0$ then $p(x) \succ 0$. In other words, if the leading coefficient is positive, then $p(x) \succ 0$. Hence, $p(x) \succ g(x)$ if $p(x) - g(x) \succ 0$.

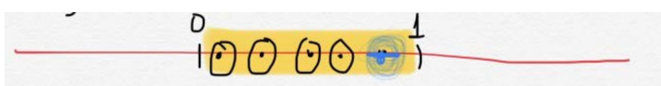
So the question is, does S satisfy the Achimedean property? Namely, if we are guven $p(x), g(x) \in S$ such that $g(x) \succ p(x) \succ 0$, does there exist an $n \in \mathbb{N}$ such that $np(x) \succ g(x)$?

Consider the following example: Let $p(x) = 10x^2 + 5$ and $g(x) = x^3 - 3x$. Both have positive leading coefficients, and thus both are "positive". Furthermore, $x^3 - 3x \succ 10x^2 + 5$ as $x^3 - 10x^2 - 3x - 5 \succ 0$. Can we find an $n \in \mathbb{N}$ such that

$$n(10x^2 + 5) \succ x^3 - 3x?$$

The answer is no, as $x^3 - 3x - 10nx^2 - 5n$ has a positive leading coefficient for all $n \in \mathbb{N}$. Therefore, S does not satisfy the Archimedian property. In some sense, $x^3 - 3x$ is "infinitely larger" than $10x^2 + 5$.

Now let's start to discuss **open sets**. A set $U \subset \mathbb{R}$ is open if for all $x \in U$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. One example of this is the set $(0, 1)$. For any number strictly between 0 and 1, you can find some $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset (0, 1)$:



Here, each dot is an x, and the circle around it is a circle of radius ϵ. Note that ϵ does **not** (and generally isn't) the same value for every x.

In the PSET, you will be asked to show that for $a, b \in \mathbb{R}$, $(-\infty, a)$, (a, b), and (b, ∞) are open sets.

But what isn't an open set? Well, consider a set like $[0, 1]$. Why isn't this an open set? For the point $1 \in [0, 1]$, for all $\epsilon > 0$ $(1 - \epsilon, 1 + \epsilon) \not\subset [0, 1]$.

In the PSET, you will be asked to show that an arbitrary union of open sets is open, and that a *finite* intersection of open sets is open. But why isn't an infinite intersection of open sets necessarily open? Here is a common counterexample.

Consider the sets of the form $U_n = (0, 1 + \frac{1}{n})$ for $n \in \mathbb{N}$. Then, $U_1 = (0, 2)$, $U_2 = (0, 1.5)$, etc.. It is clear that all of these sets are open, however

$$\bigcap_{n \in \mathbb{N}} U_n = (0, 1].$$

This set is **NOT** open, for the same reason that $[0, 1]$ wasn't open. Hence, infinite intersections of open sets may **NOT** be open.

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