Recitation 01

To-do list:

1. Proving the AM-GM inequality and introducing proof by mathematical induction.
2. Exercise 1.2.10:
   Given that $A, B \subset \mathbb{R}_{>0}$ (both bounded and nonempty), consider the set
   $$C := \{a \cdot b \mid a \in A,\ b \in B\}$$
   (where $:=$ means that we are *defining* C to be that way). Show that $\sup C = \sup A \cdot \sup B$.

The Arithmetic Mean-Geometric Mean inequality (abbreviated as AM-GM) states that for n nonnegative real numbers $x_1, \dots, x_n$,
$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \dots x_n}.$$
In the homework, y'all proved the base case of $n = 2$. Note that this inequality is true for $n = 1$, but this is a more trivial statement ($\frac{x_1}{1} \geq \sqrt[1]{x_1}$).

Normally we could try to use Standard Induction to prove this:

1. First prove the base case (which is already done).
2. Then, assume the statement is true for some k and show that this implies the statement is true for $k + 1$ (where k is an arbitrary natural number).

However, we will instead use a new method that is similar to Standard Induction:

1. Prove that the statement is true for $n = 2^k$ using induction.
2. Then, show this implies it is true for all natural numbers.

The case where $k = 1$ was done in the homework. So now, assume that the AM-GM inequality is true for some k. Then, we want to show that the inequality is true for $k + 1$:
$$\frac{x_1 + \cdots + x_{2^{k+1}}}{2^{k+1}} = \frac{(x_1 + \cdots + x_{2^k}) + (x_{2^k+1} + \cdots + x_{2^{k+1}})}{2^{k+1}}.$$
Notice that we haven't changed anything here, but we have separated the sum of 2^{k+1} terms into to sums of 2^k terms. This allows us to apply what we have assumed:
$$\begin{aligned} &\geq \frac{2^k \cdot \left(\sqrt[2^k]{x_1 \dots x_{2^k}} + \sqrt[2^k]{x_{2^k+1} \dots x_{2^{k+1}}} \right)}{2^{k+1}} \\ &= \frac{\left(\sqrt[2^k]{x_1 \dots x_{2^k}} + \sqrt[2^k]{x_{2^k+1} \dots x_{2^{k+1}}} \right)}{2}. \end{aligned}$$
Now we can apply the base case with $k = 1$:
$$\begin{aligned} &\geq \sqrt{\sqrt[2^k]{x_1 \dots x_{2^k}} \cdot \sqrt[2^k]{x_{2^k+1} \dots x_{2^{k+1}}}} \\ &= \sqrt[2^{k+1}]{x_1 \dots x_{2^{k+1}}}. \end{aligned}$$
Thus, we have shown the AM-GM inequality is true for $n = 2^k$ for all $k \in \mathbb{N}$. To show the AM-GM inequality is true for all n (not just those which are powers of 2) we want to find the nearest $m \geq n$ such that $m = 2^k$ for some

nonnegative integer k. Let

$$\mu = \frac{x_1 + \cdots + x_n}{n},$$

and let $x_{n+1} = x_{n_2} = \cdots = x_m = \mu$. Hence,

$$\begin{aligned}\mu &= \frac{x_1 + \cdots + x_n}{n} \\ &= \frac{\frac{m}{n}(x_1 + \cdots + x_n)}{m} \\ &= \frac{x_1 + \cdots + x_n + \frac{m-n}{n}(x_1 + \cdots + x_n)}{m} \\ &= \frac{x_1 + \cdots + x_n + (x_{n+1} + \cdots + x_m)}{m}.\end{aligned}$$

Applying the AM-GM inequality since $m = 2^k$:

$$\begin{aligned}\mu &\geq \sqrt[m]{x_1 \ldots x_m} \\ \mu &\geq \sqrt[m]{x_1 \ldots x_n \cdot \mu^{m-n}}.\end{aligned}$$

Moving all the μ to the left hand side and dealing with exponents, we get

$$\mu \geq \sqrt[n]{x_1 \ldots x_n}.$$

Since μ is the arithmetic mean, we are complete with our proof. $\square$

Now we will work on Exercise 1.2.10.

Step 1: We want to show that C is bounded (i.e. that the supremum exists). Given that A is bounded, there exists an α such that $a \leq \alpha$ for all $a \in A$. Similarly, there exists a β such that $b \leq \beta$ for all $b \in B$. Hence, since A and B only contain positive numbers, $ab \leq \alpha\beta$ for all $a \in A$ and all $b \in B$. Therefore, C must be bounded as C is the set of all such ab. Proving the existence of a supremum is almost always the first step in proving a statement like this one.

Step 2: Now that we know it exists, we want to show that $\sup C = \sup A \cdot \sup B$. We can do this by showing that

$$\sup C \leq \sup A \cdot \sup B \quad \text{and} \quad \sup C \geq \sup A \cdot \sup B$$

(a very common technique in analysis). It is clear that

$$\begin{cases} 0 \leq a \leq \sup A & \forall a \in A \\ 0 \leq b \leq \sup B & \forall b \in B \end{cases}.$$

Hence, $ab \leq \sup A \cdot \sup B$ for all $ab \in C$. Hence, $\sup A \cdot \sup B$ is an upper bound for C, and thus

$$\sup C \leq \sup A \cdot \sup B.$$

Now for the other direction. Fix $b \in B$ (noting of course that $b > 0$ for all $b \in B$). Then, $a \leq \frac{\sup C}{b}$ for all $a \in A$. This implies that $\frac{\sup C}{b}$ is an upper bound for A. Since $\sup A$ is the least upper bound for A, this implies that for all $b \in B$,

$$\sup A \leq \frac{\sup C}{b} \implies b \leq \frac{\sup C}{\sup A} \forall b \in B.$$

Note that $\sup A \neq 0$ as $A \neq \emptyset$, and $A \subset \mathbb{R}_{>0}$. Therefore, $\frac{\sup C}{\sup A}$ is an upper bound for B, and hence

$$\sup B \leq \frac{\sup C}{\sup A} \implies \sup A \cdot \sup B \leq \sup C.$$

Therefore, $\sup C = \sup A \cdot \sup B$. $\square$

We leave the following exercise to the student: Show that given $A, B \subset \mathbb{R}_{\geq 0}$ (such that A and B are bounded and nonempty), and C defined just as before, then $\sup C = \sup A \cdot \sup B$. The only difference between this exercise and 1.2.10 is that before we were dealing with sets of only positive numbers, and now we want to include the possibility that A and/or B contain 0.

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