18.100A: Complete Lecture Notes

Lecture 17:
Uniform Continuity and the Definition of the Derivative

Uniform Continuity

Recall 1

Recall the definition of continuity: $f : S \to \mathbb{R}$ is continuous on S if $\forall c \in S$ and $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon, c) > 0$ such that $\forall x \in S$, $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

Here, $\delta(\epsilon, c)$ denotes the fact that δ **can** depend on ϵ and c.

Example 2

Consider the function $f(x) = \frac{1}{x}$. f is continuous on $(0, 1)$.

Proof: Let $\epsilon > 0$. Choose $\delta = \min\left\{\frac{c}{2}, \frac{c^2}{2}\epsilon\right\}$. Suppose $|x - c| < \delta$. Then, $|x - c| < \frac{c}{2} \implies |x| > c - |x - c| > \frac{c}{2}$. Thus, $\frac{1}{|x|} < \frac{2}{c}$. Therefore,

$$\begin{aligned}\left|\frac{1}{x} - \frac{1}{c}\right| &= \frac{|x-c|}{|xc|} \\ &< \frac{\delta}{|x||c|} \\ &< \frac{2}{c^2}\delta \\ &\leq \frac{2}{c^2}\frac{c^2\epsilon}{2} = \epsilon.\end{aligned}$$

■

As shown in the previous example. δ depended on **both** ϵ and c.

Definition 3 (Uniformly Continuous)

Let $f : S \to \mathbb{R}$. Then, f is uniformly continuous on S if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ such that $\forall x, c \in S$,

$$|x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Remark 4. *Thus, in the definition of uniform continuity, δ only depends on ϵ!*

Example 5

The function $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

Proof: Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Then, if $x, c \in [0, 1]$ then $|x - c| < \delta$ implies that

$$|x^2 - c^2| = |x + c||x - c| \leq 2|x - c| < 2\delta = \epsilon.$$

■

However, there are of course continuous functions that are not uniformly continuous. For example, we will show that $f(x) = \frac{1}{x}$ is not uniformly continuous on (0,1), but first we consider the negation of the definition.

Negation 6 (Not Uniformly Continuous)
Let $f : S \to \mathbb{R}$. Then, f is not uniformly continuous on S if $\exists \epsilon_0 > 0$, $\forall \delta > 0$ such that $\exists x, c \in S$ with

$$|x - c| < \delta \quad \text{and} \quad |f(x) - f(c)| \geq \epsilon_0.$$

Proof: Choose $\epsilon_0 = 2$ (in fact, any $\epsilon_0 > 0$ will show that $\frac{1}{x}$ is not uniformly continuous on $(0, 1)$). Then, let $\delta > 0$. Choose $c = \min\left\{\delta, \frac{1}{2}\right\}$ and $x = \frac{c}{2}$. Then, $|x - c| = \frac{c}{2} \leq \frac{\delta}{2} < \delta$ and

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{2}{c} - \frac{1}{c}\right| = \frac{1}{c} \geq \frac{1}{\frac{1}{2}} = 2.$$

□

Theorem 7
Let $f : [a, b] \to \mathbb{R}$. Then, f is continuous if and only if f is uniformly continuous.

Proof: ($\Longleftarrow$) This direction is left as an exercise to the reader.

($\Longrightarrow$) Suppose f is continuous and assume for the sake of contradiction that f is *not* uniformly continuous. Then, $\exists \epsilon_0 > 0$ such that for all $n \in \mathbb{N}$, $\exists x_n, c_n \in [a, b]$ such that

$$|x_n - c_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(c_n)| > \epsilon_0.$$

By Bolzano-Weierstrass, $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in [a, b]$ such that $\lim_{k\to\infty} x_{n_k} = x$. Similarly, by Bolzano-Weierstrass, $\exists$ a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ and $c \in [a, b]$ such that $\lim_{k\to\infty} c_{n_k} = c$. Note that subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ satisfies $\lim_{j\to\infty} x_{n_{k_j}} = x$.

Then,

$$|x - c| = \lim_{j\to\infty} |x_{n_{k_j}} - c_{n_{k_j}}| \leq \lim_{j\to\infty} \frac{1}{n_{k_j}} - 0.$$

Thus, $x = c$. But, since f is continuous at c,

$$0 = |f(c) - f(c)| = \lim_{j\to\infty} |f(x_{n_{k_j}}) - f(c_{n_{k_j}})| \geq \epsilon_0.$$

This is a contradiction. □

Derivative

Definition 8
Let I be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that f is differentiable at c if the limit

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

exists.

Notation 9

If f is differentiable at c, we write

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Furthermore, if f is differentiable at every $c \in I$, we write f' or $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the function $f'(x)$.

Example 10

Consider the function $f(x) = ax + b$. Then, for all $c \in \mathbb{R}$, $f'(c) = a$.

Proof: This follows as

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{ax + b - (ac + b)}{x - c} = a \lim_{x \to c} \frac{x - c}{x - c} = \lim_{x \to c} a = a.$$

$\blacksquare$

Example 11 (The Power Rule)

For all $n \in \mathbb{N}$, if $f(x) = \alpha x^n$, then for all $c \in \mathbb{R}$,

$$f'(c) = \alpha n c^{n-1}.$$

Proof: We note that for all $n \in \mathbb{N}$,

$$(x - c) \sum_{j=0}^{n-1} x^{n-1-j} c^j = \sum_{j=0}^{n-1} x^{n-j} c^j - \sum_{j=0}^{n-1} x^{n-1-j} c^{j+1}.$$

Letting $\ell = j + 1$, we obtain

$$\begin{aligned} (x - c) \sum_{j=0}^{n-1} x^{n-1-j} c^j &= \sum_{j=0}^{n-1} x^{n-j} c^j - \sum_{\ell=1}^{n} x^{n-\ell} c^\ell \\ &= x^{n-0} c^0 - x^{n-n} c^n \\ &= x^n - c^n. \end{aligned}$$

Therefore,

$$\lim_{x \to c} \frac{\alpha x^n - \alpha c^n}{x - c} = \alpha \lim_{x \to c} \sum_{j=0}^{n-1} x^{n-1-j} c^j = \alpha \sum_{j=0}^{n-1} c^{n-1-j} c^j = \alpha n c^{n-1}.$$

$\blacksquare$