Exercises given with a numbering are from *Basic Analysis: Introduction to Real Analysis (Vol I)* by J. Lebl.

Reading Sections 3.3, 3.4, 3.5

Exercises

1. Exercise 3.3.11
2. Exercise 3.4.3
3. Exercise 3.4.8
4. Let $S \subset \mathbb{R}$. We say that $f : S \to \mathbb{R}$ is *Lipschitz continuous* on S if there exists $L \geq 0$ such that for all $x, y \in S$,

$$|f(x) - f(y)| \leq L|x - y|.$$

   Prove that if $f : S \to \mathbb{R}$ is Lipschitz continuous on S then f is uniformly continuous on S.

5. (a) Prove that $f(x) = \cos x$ is Lipschitz continuous on $\mathbb{R}$.
   (b) Prove that $f(x) = x^{1/3}$ is uniformly continuous on $[0, 1]$ and is not Lipschitz continuous on $[0, 1]$.
6. Let $R \in \mathbb{R}$, and let $f : [R, \infty) \to \mathbb{R}$. We say that $f(x)$ *converges to L as $x \to \infty$* if for every $\epsilon > 0$ there exists $M \geq R$ such that for all $x \geq M$ we have $|f(x) - L| < \epsilon$. We write $f(x) \to L$ as $x \to \infty$ or

$$\lim_{x \to \infty} f(x) = L.$$

   [A similar definition can be formulated for limits as $x \to -\infty$ but we will not do so here.]

   (a) Prove that

$$\lim_{x \to \infty} \frac{x^2}{x^2 + 1} = 1.$$

   (b) Prove that

$$\lim_{x \to \infty} \sin x$$

   does not exist.