Exercises given with a numbering are from *Basic Analysis: Introduction to Real Analysis (Vol I)* by J. Lebl.

Reading Sections 2.1, 2.2

Exercises

1. We say a set $F \subset \mathbb{R}$ is *closed* if its complement $F^c := \mathbb{R} \setminus F$ is open (see Assignment 3 for a discussion of open sets). Since $\varnothing$ and $\mathbb{R}$ are open, it follows that $\varnothing$ and $\mathbb{R}$ are closed as well.
   (a) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $[a, b]$ is closed.
   (b) Is the set $\mathbb{Z} \subset \mathbb{R}$ closed? Provide a proof to substantiate your claim.
   (c) Is the set of rationals $\mathbb{Q} \subset \mathbb{R}$ closed? Provide a proof to substantiate your claim.
2. (a) Let Λ be a set (not necessarily a subset of $\mathbb{R}$), and for each $\lambda \in \Lambda$, let $F_\lambda \subset \mathbb{R}$. Prove that if F_λ is closed for all $\lambda \in \Lambda$ then the set
   $$\bigcap_{\lambda \in \Lambda} F_\lambda = \{x \in \mathbb{R} : x \in F_\lambda \text{ for all } \lambda \in \Lambda\}$$
   is closed.
   (b) Let $n \in \mathbb{N}$, and let $F_1, \ldots F_n \subset \mathbb{R}$. Prove that if $F_1, \ldots, F_n$ are closed then the set $\cup_{m=1}^n F_m$ is closed.
3. Let $F \subset \mathbb{R}$ be a closed set, and let $\{x_n\}$ be a sequence of elements of F converging to $x \in \mathbb{R}$. Prove that $x \in F$.
   Hint: Assume that $x \in F^c$ and arrive at a contradiction.
4. Exercise 2.2.3
5. Exercise 2.2.5
6. Let $A \subset \mathbb{R}$ be bounded above, and let a_0 be an upper bound for A. Prove that $a_0 = \sup A$ if and only if there exists a sequence $\{a_n\}$ of elements of A such that $\lim_{n\to\infty} a_n = a_0$.
   Hint: By Assignment 3, if $a_0 = \sup A$ then for all $n \in \mathbb{N}$ there exists $a_n \in A$ such that
   $$a_0 - \frac{1}{n} < a_n \leq a_0.$$
7. Let $E \subset \mathbb{R}$ be a nonempty set of real numbers. We say $x \in \mathbb{R}$ is a *cluster point* of E if for every $\epsilon > 0$
   $$(x - \epsilon, x + \epsilon) \cap E \setminus \{x\} \neq \varnothing.$$
   Said less formally, x is a cluster point of E if every interval containing x contains at least one element of E other than x.

(a) Prove that x is a cluster point of E if and only if there exists a sequence $\{x_n\}$ of elements of $E \setminus \{x\}$ such that $\lim_{n\to\infty} x_n = x$.
Hint: If x is a cluster point of E, then for all $n \in \mathbb{N}$ there exists $x_n \in E$ with $x_n \neq x$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}.$$

(b) Prove that the set of all cluster points of E is closed.