Exercises given with a numbering are from *Basic Analysis: Introduction to Real Analysis (Vol I)* by J. Lebl.

Reading Sections 4.3, *The Riemann Integral* lecture notes

Exercises

1. Prove that the polynomial equation $\frac{x^{1121}}{1121} + \frac{x^{2021}}{2021} + x + 1 = 0$ has exactly one real root.
2. Compute the fourth Taylor polynomial for:
   (a) $f(x) = \sin x$ at $x = 0$.
   (b) $f(x) = \frac{1}{1-x}$ at $x = -1$.
3. Compute:
   (a)
   $$\lim_{x\to 0} \frac{x - \sin x}{x^3}$$
   (b)
   $$\lim_{x\to \frac{\pi}{2}} \frac{1 - \sin x}{\left(x - \frac{\pi}{2}\right)^2}$$
4. Suppose that $f : (a, b) \to \mathbb{R}$ is three times continuously differentiable, $c \in (a, b)$, $f'(c) = f''(c) = 0$ and $f'''(c) > 0$. Prove that f has a neither a local maximum nor a local minimum at c.
5. Let $a < b$, and define a sequence of tagged partitions by
   $$\underline{x}^{(r)} := \left\{a + (b-a)\frac{k}{r} : k = 0, \dots, r\right\},$$
   $$\underline{\xi}^{(r)} := \left\{a + (b-a)\frac{k}{r} : k = 1, \dots, r\right\}.$$
   (a) Compute $\|\underline{x}^{(r)}\|$.
   (b) Let $f(x) = \alpha x + \beta$. Prove that
   $$\lim_{r\to\infty} S_f\left(\underline{x}^{(r)}, \underline{\xi}^{(r)}\right) = \alpha\frac{b^2 - a^2}{2} + \beta(b-a).$$
   You may not use the fundamental theorem of calculus.