Chapters and exercises given with a numbering are from *Basic Analysis: Introduction to Real Analysis (Vol I)* by J. Lebl.

Reading Section 0.3

Exercises

1. Exercise 0.3.6
2. Exercise 0.3.11
3. Exercise 0.3.12
4. Exercise 0.3.15
5. Exercise 0.3.19
6. In this exercise, you will prove that

$$|\{q \in \mathbb{Q} : q > 0\}| = |\mathbb{N}|.$$

In what follows, we will use the following theorem without proof:

Theorem. *Let $q \in \mathbb{Q}$ with $q > 0$. Then*

1) if $q \in \mathbb{N}$ and $q \neq 1$, then there exists unique prime numbers $p_1 < p_2 < \cdots < p_N$ and unique exponents $r_1, \ldots, r_N \in \mathbb{N}$ such that

$$q = p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}, \tag{†}$$

2) if $q \notin \mathbb{N}$, then there exist unique prime numbers $p_1 < p_2 < \ldots < p_N$, $q_1 < q_2 < \cdots < q_M$ with $p_i \neq q_j$ for all $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots M\}$, and unique exponents $r_1, \ldots, r_N, s_1, \ldots s_M \in \mathbb{N}$ such that

$$q = \frac{p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}}{q_1^{s_1} q_2^{s_2} \cdots q_M^{s_M}}.. \tag{‡}$$

Define $f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ as follows: $f(1) = 1$, if $q \in \mathbb{N}\backslash\{1\}$ is given by (†), then

$$f(q) = p_1^{2r_1} \cdots p_N^{2r_N},$$

and if $q \in \mathbb{Q}\backslash\mathbb{N}$ is given by (‡), then

$$f(q) = p_1^{2r_1} \cdots p_N^{2r_N} q_1^{2s_1 - 1} \cdots q_M^{2s_M - 1}.$$

(a) Compute $f(4/15)$. Find q such that $f(q) = 108$.

(b) Use the **Theorem** to prove that f is a bijection.