18.075 Solutions to Practice Test 3 for Exam 3

(I.) (a) The Frobenius series for $J_0(x)$ is:

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

$$\Rightarrow J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \frac{2k}{2^{2k}} x^{2k-1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \frac{k}{2^{2k-1}} x^{2k-1}$$

$$= -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!\,(k-1)!} \left(\frac{x}{2}\right)^{2k-1} \underset{=}{\overset{k=m+1}{=}} \underbrace{-\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+1)!} \left(\frac{x}{2}\right)^{2m+1}}_{\text{Frobenius series for } J_1(x)!}$$

$$= -J_1(x)$$

(b) $$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} \left(\frac{x}{2}\right)^{2k+1} \Rightarrow x J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} \frac{x^{2k+2}}{2^{2k+1}}$$

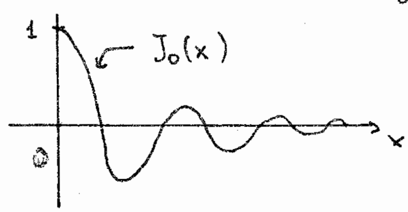
$$\Rightarrow \frac{d}{dx}[x J_1(x)] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!} \cancel{2}(k+1) \frac{\overbrace{x^{2k+1}}^{x\cdot x^{2k}}}{2^{2k+\cancel{1}}}$$

$$= x \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\cdot k!} \left(\frac{x}{2}\right)^{2k} = x J_0(x).$$

(II.) (a) $$\int_0^1 dx\; J_0(x)\, J_1(x) = \int_0^1 dx\; J_0(x) \underbrace{\left[-\frac{d}{dx} J_0(x)\right]}_{\text{from part (a) of Prob. I.}} = -\frac{1}{2} \int_0^1 dx \cdot \left[\frac{d}{dx} J_0(x)^2\right]$$

$$= -\frac{1}{2} J_0(x)^2 \Big|_{x=0}^{1} = -\frac{1}{2} \left[J_0(1)^2 - \underbrace{J_0(0)^2}_{=1}\right] = \frac{1}{2} \left[1 - J_0(1)^2\right]$$

The function $J_0(x)$ starts with the value $J_0(0) = 1$ at $x=0$ and then oscillates to <u>smaller</u> values tending to 0 as $x \to \infty$:



Hence, $1 - J_0(1)^2 > 0$: the result of integration is <u>positive</u>.

(b) $\displaystyle\int_0^1 dx\; x^3\, J_0(x) = \int_0^1 dx \cdot x^2\, [x J_0(x)] = \int_0^1 dx \cdot x^2 \underbrace{\frac{d}{dx}[x J_1(x)]}_{\text{from part (b) of Prob. I}}$

(Integration by parts:)

$$= x^2 \cdot x J_1(x)\Big|_0^1 - 2\int_0^1 dx\; x \cdot x J_1(x) = J_1(1) - 2\int_0^1 dx \cdot x^2 J_1(x)$$

$$= J_1(1) + 2\int_0^1 dx\; x^2 \underbrace{\left[\frac{d}{dx} J_0(x)\right]}_{\substack{\text{from part (a)} \\ \text{of Prob. I}}} = J_1(1) + 2\,x^2 J_0(x)\Big|_0^1 - 4\int_0^1 dx\; x J_0(x)$$

$$= J_1(1) + 2J_0(1) - 4\int_0^1 dx\; \underbrace{x J_0(x)}_{= \frac{d}{dx}[x J_1(x)],\ \text{from part (b) of Prob. I}} = J_1(1) + 2J_0(1) - 4\, x J_1(x)\Big|_0^1$$

$$= J_1(1) + 2J_0(1) - 4J_1(1) = -3J_1(1) + 2J_0(1)$$

(III.) We solve the ODE

$$y''(t) + \frac{1}{t}\, y'(t) - y(t) = 0,$$

with the condition $\quad y(0) = 1.$

The ODE is written as: $\quad t^2 y''(t) + t\, y'(t) - t^2 y(t) = 0.$

The general solution of this ODE, which is of the form $\; t^2 y'' + t y' - (t^2 + p^2) y = 0$ with $\underline{p = 0}$,

is

$$y(t) = c_1 \cdot I_0(t) + c_2 K_0(t), \qquad I_0, K_0: \underline{\text{modified}} \text{ Bessel functions.}$$

<u>Recall</u>: $K_0(t)$ "<u>blows up</u>" at $t = 0$.

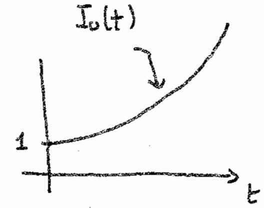
By requiring that $y(0) = 1$: finite we <u>must</u> <u>impose</u> $\boxed{c_2 = 0}$.

Hence, $\; y(t) = c_1 I_0(t)\;$; $\; y(0) = 1 \Rightarrow c_1 \cdot \underbrace{I_0(0)}_{1} = 1 \Rightarrow \boxed{c_1 = 1}$

Solution: $\; y(t) = I_0(t)$.

Recall (again!) that $I_0(t)$ "blows up" as $t \to \infty$.

This model indeed describes exponential growth in time.



(IV)
$$\begin{cases} \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + k^2 z = 0 \,, & -a \le x \le a, \; -b \le y \le b \\ z(\pm a, y) = 0, \; z(x, \pm b) = 0 \end{cases}$$

(a) Replace $z(x,y)$ by the product

$$z(x,y) = X(x)\, Y(y) \,,$$

which is what we called "separation of variables" in class.

$$\frac{\partial^2 z}{\partial x^2} = Y(y) \frac{d^2 X}{dx^2} \,, \qquad \frac{\partial^2 z}{\partial y^2} = X(x) \frac{d^2 Y}{dy^2} \,.$$

The PDE becomes

$$Y \cdot X'' + X Y'' + k^2 X Y = 0$$

$$\xrightarrow{\cdot \frac{1}{XY}} \quad \frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0. \quad \Longrightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = -k^2 = \text{const.}$$

But $\frac{X''}{X}$ is <u>only</u> a function of x and $\frac{Y''}{Y}$ is <u>only</u> a function of y.

The only way to make their sum equal to a constant is to have each of them equal to a constant:

$$\frac{X''}{X} = -p^2 = \text{const.} \,, \qquad \frac{Y''}{Y} = -q^2 = \text{const.}$$

Of course, we must have $p^2 + q^2 = k^2$.

ODEs for X, Y: $\quad X'' + p^2 X = 0 \,, \qquad Y'' + q^2 Y = 0 \,.$

Boundary conditions: $\quad X(0) = 0 = X(a) \,, \qquad Y(0) = 0 = Y(b).$

(b), Solve the boundary-value problem for $X(x)$:

$$\begin{cases} X'' + p^2 X = 0 , & -a \le x \le a \\ X(-a) = 0 = X(a) \end{cases}$$

$X(x) = A\cos(px) + B\sin(px)$:

$$\left.\begin{array}{l} X(a)=0 \rightarrow \quad A\cos(pa) + B\sin(pa) = 0 \\ X(-a)=0 \rightarrow \quad A\cos(pa) - B\sin(pa) = 0 \end{array}\right\}$$

This system of equations has non-trivial solutions <u>only if</u>

$$\begin{vmatrix} \cos(pa) & \sin(pa) \\ \cos(pa) & -\sin(pa) \end{vmatrix} = 0 \Rightarrow \sin(2pa) = 0 \Rightarrow 2pa = n\pi \Rightarrow \boxed{p = \frac{n\pi}{2a}} ,$$

$\underline{n = 1, 2, \dots}$

[For $n=0$, $p=0$ and $X \equiv 0$: trivial.]

- Boundary-value problem for $Y(y)$:

$$\begin{cases} Y'' + q^2 Y = 0 , & -b \le y \le b. \\ Y(-b) = 0 = Y(b) \end{cases}$$

$Y(y) = C\cos(qy) + D\sin(qy)$:

$$\left.\begin{array}{l} Y(b)=0 \Rightarrow \quad C\cos(qb) + D\sin(qb) = 0 \\ Y(-b)=0 \Rightarrow \quad C\cos(qb) - D\sin(qb) = 0 \end{array}\right\}$$

This system of equations has non-trivial solutions <u>only if</u>

$$\begin{vmatrix} \cos(qb) & \sin(qb) \\ \cos(qb) & -\sin(qb) \end{vmatrix} = 0 \Rightarrow \sin(2qb) = 0 \Rightarrow 2qb = m\pi \Rightarrow \boxed{q = \frac{m\pi}{2b}} ,$$

$\underline{m = 1, 2, \dots}$

[For $m=0 \rightarrow q=0$ and $Y \equiv 0$: trivial.]

It follows that $$k^2 = p^2 + q^2 = \left(\frac{n\pi}{2a}\right)^2 + \left(\frac{m\pi}{2b}\right)^2 = \frac{\omega^2}{c^2}$$

$$\Rightarrow \omega = c\sqrt{\left(\frac{n\pi}{2a}\right)^2 + \left(\frac{m\pi}{2b}\right)^2}$$: characteristic frequencies of membrane.

(V.) $$x^2 y'' + x(x^2-\lambda)\, y' + (x^2+\lambda)\, y = 0$$

(a) ODE : $$y'' + \frac{1}{x}(-\lambda + x^2)\, y' + \frac{1}{x^2}(\lambda + x^2)\, y = 0 \;;$$

$R(x) = 1, \quad P(x) = -\lambda + x^2, \quad Q(x) = \lambda + x^2.$

(b) Indicial equation: $f(s) = s(s-1) + P_0 s + Q_0 = 0 \;;\; P_0 = -\lambda,\ Q_0 = \lambda$

$\Rightarrow\ f(s) = s(s-1) - \lambda s + \lambda = s(s-1) - \lambda(s-1) = (s-\lambda)(s-1) = 0$

$\Rightarrow\ \{ s = \lambda,\ s = 1 \}$. If $s_1 \geq s_2$ then $\begin{cases} s_1 = \lambda,\ s_2 = 1 & \text{if } \underline{\lambda > 1} \\ s_1 = 1,\ s_2 = \lambda & \text{if } \underline{\lambda < 1}. \end{cases}$

(c) If λ is <u>not</u> an integer, then $s_1 - s_2 = |1-\lambda| \neq$ integer.

It follows that in this case the method of Frobenius yields <u>2</u> independent solutions.

If $\lambda = 1$, then $s_1 = s_2 = 1$ and the method of Frobenius yields <u>1</u> (independent) solution.

(d) Suppose that $\lambda > 1$, with $\lambda = m$: integer.

Then $s_1 = \lambda = m$ and $s_2 = 1 \Rightarrow \underline{s_1 - s_2 = m-1 : \text{integer}}$

$g_n(s) = P_n (s-n)(s-n-1) + R_n (s-n) + Q_n, \quad n \geq 1.$

Clearly $g_n(s) \equiv 0$ <u>unless</u> $\underline{n = 2}$.

$g_2(s) = (s-2)(s-3) + 1 = s^2 - 5s + 7$

$g_2(s_2 + k) = g_2(1+k) = (k-1)(k-2) + 1.$

$\boxed{\text{Recursive formula for } s=s_2=1:}$

$$f(\overset{1}{\overset{\shortparallel}{s_2}}+k)\cdot A_k = -\sum_{n=1}^{k} g_n(s_2+k)\cdot A_{k-n}, \qquad k=1,2,\ldots$$

$$\Longrightarrow \quad k(1+k-m)\, A_k = -\sum_{n=1}^{k} g_n(s_2+k)\cdot A_{k-n}$$

$\underline{k=1}$: $\quad (2-m)\, A_1 = 0$

$\underline{k\geq 2}$: $\quad k(k+1-m)\, A_k = -g_2(1+k)\cdot A_{k-2}$

$$\Longrightarrow \quad \boxed{k(k+1-m)\, A_k = -(k^2-k+1)\, A_{k-2}}, \qquad k=2,3,\ldots$$

(e) Assume that $\lambda = m > 1$ $\underline{\text{and}}$ $m=2\ell$: $\overset{\text{even}}{\text{integer}}$, $\ell = 1,2,\ldots$

We check the recursive formula for $k = s_1 - s_2 = m-1$.

- Consider $\underline{\ell=1}$, i.e., $\underline{m=2}$.

$\underline{k=m-1=1}$: $\quad 0\cdot A_1 = 0 \Longrightarrow A_1$: arbitrary ($A_0$: also arbitrary)

Hence, for $\underline{\lambda=m=2}$, the Frobenius method gives $\underline{\text{2 independent solutions.}}$

- Now consider $\ell \geq 2$, i.e., $m=4,6,\ldots$

$\underline{k=m-1}$: $\quad 0\cdot A_{m-1} = -\underbrace{[(m-1)^2-(m-1)+1]}_{\neq 0}\cdot A_{m-3} = -[(m-1)^2-(m-1)+1]\cdot A_{2\ell-3}$

$=2\ell-1$

$\underline{k=1}$: $\quad (2-m)\, A_1 = 0 \Longrightarrow A_1 = 0$

$\underline{k=2}$: $\quad 2(3-m)\, A_2 = -(2^2-2+1)\, A_0 \neq 0 \Longrightarrow A_2 = -\dfrac{3}{2(3-m)}\, A_0$

$\underline{k=3}$: $\quad 3\cdot(4-m)\, A_3 = -(3^2-3+1)\, A_1 = 0 \Longrightarrow A_3 = 0$.

$\underline{\text{and so on.}}$

It follows that $\quad A_k = 0 \quad$ for $\quad k = 1,3,5,\ldots, 2\ell-3$, i.e., coefficients with odd index smaller than $2\ell-1=m-1$ are zero.

The recursive relation for $k=m-1$ gives

$k=m-1=2\ell-1$: $\quad 0\cdot A_{2\ell-1} = -\left[(m-1)^2-(m-1)+1\right]\cdot 0 = 0$

$$\Longrightarrow \quad A_{2\ell-1} : \text{arbitrary} \qquad (A_0 : \text{also arbitrary})$$

It follows that for $m=2\ell$ the Frobenius method gives 2 indep. solutions

(VI) ODE : $\quad x^2y'' + x(x^2-\lambda)y' + (x^2+\lambda)y = 0$

Compare with the form

$$x^2y'' + x\left[(1-2A) + 2rBx^r\right]y' + \left[A^2-p^2s^2+s^2C^2x^{2s}-rB(2A-r)x^r+r^2B^2x^{2r}\right]y=0$$

that was given in class. The latter equation has solution

$$y(x) = g(x)\,Z_p[f(x)] \quad ; \quad g(x) = x^A e^{-Bx^r}, \qquad f(x) = Cx^s.$$

(Z_p: Bessel fcn of order p.)

Coeff. of y'

$1-2A = -\lambda \Leftrightarrow A = \frac{1+\lambda}{2}$

$r=2$; $\quad 2rB=1 \Leftrightarrow B=\frac{1}{4}$.

Coeff. of y

The coefficient of $x^r = x^2$ should be

$$-rB(2A-r) = -2\,\tfrac{1}{4}(1+\lambda-2) = -\tfrac{1}{2}(\lambda-1) = 1$$

$$\Leftrightarrow \boxed{\lambda=-1}$$

For $\lambda=-1$, we need $2s=2r \Rightarrow s=r=2$,

- coefficient of x^4 must vanish:

$$s^2C^2+r^2B^2=0 \Leftrightarrow s^2C^2=-\tfrac{1}{4} \Leftrightarrow sC=\tfrac{i}{2}$$

$$\Longrightarrow \boxed{C=i/4}$$

$A^2-p^2s^2=\lambda=-1 \Leftrightarrow p^2s^2=1 \Leftrightarrow p=\frac{1}{2}$.

The solution is

$$y(x) = e^{-x^2/4} \; Z_{1/2}\left(\tfrac{i}{4} x^2\right)$$

$$= e^{-x^2/4} \; \left[c_1 I_{1/2}\left(x^2/4\right) + c_2 I_{-1/2}\left(x^2/4\right)\right]$$

modified Bessel fcns