Case I $\quad \int_{-\infty}^{\infty} dx\, \frac{P_N(x)}{Q_M(x)}, \quad M \geq N+2$ (so it converges) $\quad$ ex $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Definition: $f(z)$, $z = re^{i\theta}$, tends to 0 uniformly for $\theta_1 \leq \theta \leq \theta_2$ as $r \to \infty$
if $\begin{cases} |f(z)| \leq K(r), & \theta_1 \leq \theta \leq \theta_2 \\ \text{and } K(r) \to 0, & r \to \infty \end{cases}$

Theorem 1: If $zf(z) \to 0$ uniformly as $|z| = R \to +\infty$, $\quad z = Re^{i\theta}$
then $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$, $\quad C_R$ = circular arc (any), $|z| = R$, $\theta_1 \leq \theta \leq \theta_2$

$$\text{ex } |zf(z)| = \frac{|z|}{\underbrace{|z^2+1|}_{\geq ||z^2|-1|}} = \frac{R}{|R^2 e^{i2\theta}+1|} \leq \frac{R}{R^2-1} \to 0 \quad (R \to \infty)$$

$|z_1 + z_2| \geq ||z_1| - |z_2||$

Case II $\quad \int_{-\infty}^{\infty} dx\, e^{i\alpha x} \frac{P_N(x)}{Q_M(x)} \quad \alpha$: real

Theorem 2: If $f(z) \to 0$ uniformly as $|z| \to \infty$, then

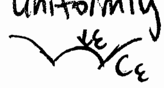
(i) $\alpha > 0$, $\lim_{R\to\infty} \int_{C_u} e^{i\alpha z} f(z)\,dz = 0$

(ii) $\alpha < 0$, $\lim_{R\to\infty} \int_{C_{lo}} e^{i\alpha z} f(z)\,dz = 0$

(iii) $\alpha > 0$, $\lim_{R\to\infty} \int_{C_l} e^{\alpha z} f(z)\,dz = 0$

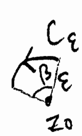
(iv) $\alpha < 0$, $\lim_{R\to\infty} \int_{C_r} e^{\alpha z} f(z)\,dz = 0$

Theorem 3: If $(z - z_0) f(z) \to 0$ uniformly as $|z - z_0| \equiv \varepsilon \to 0$, then
$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz = 0$



Digression: $F(z) \to 0$ uniformly when $|z - z_0| \equiv \varepsilon \to 0$, if $\begin{cases} |F(z)| < A(\varepsilon) \\ A(\varepsilon) \to 0, \ \varepsilon \to 0 \end{cases}$

Theorem 4: If z_0: simple pole of $f(z)$, then $\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz = i\beta \operatorname{Res}_{z=z_0} f(z)$



Theorem 3 is a special case of Theorem 4.