<u>ex</u> $f(z) = \dfrac{z^3-2z+1}{z^5+2z^3+z} = \dfrac{(z^3-z)-(z-1)}{(z^4+2z^2+1)z} = \dfrac{(z-1)(z(z+1)-1)}{z[(z^4+z^2)+(z^2+1)]} = \dfrac{(z-1)(z^2+z-1)}{z\underbrace{(z^2+1)^2}_{(z+i)(z-i)}}$

$$= \frac{(z-1)(z^2+z-1)}{z(z+i)^2(z-i)^2}$$

$z_0 = 0$: $m=1$ $\quad zf(z) = \dfrac{(z-1)(z^2+z-1)}{(z+i)^2(z-i)^2}$; analytic, $\neq 0$ → simple pole

$z_0 = -i$: $\quad (z-z_0)^2 f(z) \begin{cases} \text{analytic} \\ \text{not zero as } z \to z_0 \end{cases}$ → double pole (same for i.)

possible (isolated) singularities: $0, \pm i$: poles

<u>ex</u> $f(z) = \dfrac{z}{\sin z}$ $\qquad$ singularities: $z: n\pi$, n: integer

<u>$n=0$:</u> $z_0 = 0$ $\qquad \dfrac{z}{z - \frac{z^3}{3!} + \dots + (-1)^n \frac{z^{2n+1}}{(2n+1)!}}$; analytic in $0 \le |z - z_0| < \delta$

take out z → $\qquad$ z_0 is removable.

<u>$n \neq 0$:</u> $z_0 = n\pi$ $\qquad z - n\pi = z - z_0 = t \to z = t + n\pi$

$$f(z) = \frac{t+n\pi}{\underbrace{\sin(t+n\pi)}_{(-1)^n \sin t}} = (-1)^n\,(t+n\pi)\,\frac{1}{t - \frac{t^3}{3!} + \dots + (-1)^k \frac{t^{2k+1}}{(2k+1)!}} = \frac{1}{t}\left[\frac{(-1)^n(t+n\pi)}{1 + t^2/3! + \dots}\right]$$

simple pole $\qquad$ $g(t)$: analytic for $0 \le |t| < \delta$

$t=0$: simple pole → $z = n\pi$: simple pole

<u>ex</u> $f(z) = \dfrac{Ln(1+z)}{z}$ → branch point at $z=-1$

→ singular point at $z=0$; simple pole?

because $zf(z) = Ln(1+z)$ analytic at $z_0 = 0$

$zf(z)\big|_{z=0} = 0$. no.

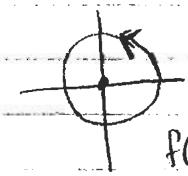
$z_0 = 0$: $\dfrac{Ln(1+z)}{z} = \dfrac{z - z^2/2 + z^3/3 + \dots}{z} = 1 - \dfrac{z}{2} + \dfrac{z^2}{3} + \dots$ $\quad$ Taylor: analytic

∴ $z=0$: removable singularity, only if $Ln(1+z)\big|_{z=0} = 0$

simple pole if $Ln(1+z)\big|_{z=0} = im2\pi$

ex $f(z) = \frac{\ln(z)}{z}$; possible singularity: $z_0 = 0$

· is $z_0 = 0$ isolated? $\underbrace{\frac{1}{z}}_{\text{has a simple pole at } z_0 = 0} \cdot \underbrace{\ln z}_{\text{has a branch point at } z_0 = 0}$ → dominant role



$f(z)$ becomes multiple valued, you can't make Laurent series.

$f(z)$ is multiple-valued around $z_0 = 0$, hence, $f(z)$ cannot admit a Laurent series in $0 < |z - z_0| < \delta$.

single valued