18.075 Solutions to Practice Test 2 for Exam 3

(I.) The Frobenius series for the Bessel function $J_p(x)$ is

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(p+k)} \left(\frac{x}{2}\right)^{2k+p} = 2^{-p} \sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k}{2^{2k}\,k!\,\Gamma(p+k)} \quad x^{2k}}_{A_k(x)}$$

We apply the ratio test for the whole term $A_k(x)$:

$$L_k(x) \equiv \left| \frac{A_{k+1}(x)}{A_k(x)} \right| = \left| \frac{\dfrac{(-1)^{k+1}}{2^{2k+2}(k+1)!\,\Gamma(p+k+1)} x^{2k+2}}{\dfrac{(-1)^k}{2^{2k}\,k!\,\Gamma(p+k)} x^{2k}} \right| = \frac{1}{4} \frac{k!}{(k+1)!} \frac{\Gamma(p+k)}{\Gamma(p+k+1)} |x^2|$$

$$= \frac{\cancel{k!}}{(k+1)\cancel{k!}} \frac{\cancel{\Gamma(p+k)}}{(p+k)\cdot\cancel{\Gamma(p+k)}} |x^2| = \frac{1}{k+1} \frac{1}{p+k} |x^2| \xrightarrow[k\to\infty]{} 0 \cdot |x^2| = 0 < 1 \quad \underline{\text{for all } x},$$

where we used the property $\Gamma(x+1) = x\,\Gamma(x)$ for $x = p+k$.

Hence $L(x) = \lim_{k\to\infty} L_k(x) = 0 < 1$ for all $x \implies R = \infty$ (the series converges everywhere)

(II.) ODE: $(\sin x)^2 y'' + x\,y' + (1-\cos x)\,y = 0$

$$\implies y'' + \underbrace{\frac{x}{(\sin x)^2}}_{a_1(x)} y' + \underbrace{\frac{1-\cos x}{(\sin x)^2}}_{a_2(x)} y = 0$$

(1)

Possible singularities: points where $a_1(z)$ or $a_2(z)$ is NOT analytic, e.g.,

$$\sin x = 0 \iff x = x_n = n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

$a_1(z) = \dfrac{1}{\sin z}\left(\dfrac{z}{\sin z}\right)$, where $\dfrac{z}{\sin z}$ is analytic at $z=0$ but NOT at $z = n\pi$, $n \neq 0$,

and $\dfrac{1}{\sin z}$ is NOT analytic at $z = n\pi$, all n.

Hence, $a_1(z)$ is NOT analytic at $z = x_n = n\pi$, $n = 0, \pm 1, \pm 2, \dots$

$\implies z = x_n$ are singular points of the ODE

② Take $x_0 = 0$; this is a singular point of the ODE (for $n=0$).

- $(z-x_0)\, a_1(z) = z\, a_1(z) = \dfrac{z^2}{(\sin z)^2} = \left(\dfrac{z}{\sin z}\right)^2$: analytic at $z = x_0 = 0$.
- $(z-x_0)^2 a_2(z) = z^2 a_2(z) = z^2 \dfrac{1-\cos z}{(\sin z)^2}$

We need to check whether the RHS is analytic at $z=0$. Hence, we need to expand the RHS around $z=0$. We expand numerator and denominator separately:

$$1-\cos z = 1-\left(1 - z^2/2! + z^4/4! - \dots\right) = z^2/2! - z^4/4! + \dots = \frac{z^2}{2!}\left(1 - \frac{2!}{4!} z^2 + \dots\right),$$

$$\sin z = z - z^3/3! + z^5/5! - \dots = z\left(1 - z^2/3! + z^4/5! - \dots\right)$$

$$\rightarrow (\sin z)^2 = z^2\left(1 - z^2/3! + z^4/5! - \dots\right)^2.$$

It follows that
$$z^2 a_2(z) = \cancel{z^2} \cdot \frac{\cancel{z^2}/2!\ \overbrace{\left(1 - \frac{2!}{4!} z^2 + \dots\right)}^{\text{Taylor series, } \neq 0 \text{ at } z=0}}{\cancel{z^2}\ \underbrace{\left(1 - z^2/3! + z^4/5! - \dots\right)^2}_{\text{Taylor series, } \neq 0 \text{ at } z=0}}$$

Hence, $z^2 a_2(z)$ is analytic at $z=0$ (Note: $a_2(z)$ is analytic at $z=0$, to start with!)

Hence, $z=0$ is a <u>regular</u> singular point of this ODE.

Ⅲ. ODE: $xy'' + y = 0$

① In order to classify the point $x_0 = 0$, we put this ODE in the form

$y'' + a_1(x) y' + a_2(x) y = 0$; $a_1(x) = 0$, $a_2(x) = \frac{1}{x}$: NOT analytic at $x_0 = 0$

Hence, $x_0 = 0$ is a singular point. Since $x^2 a_2(x) = x$: analytic,

$x_0 = 0$ is a <u>regular</u> singular point.

② Now we put the ODE in the canonical form, $R(x) y'' + \frac{1}{x} P(x) y' + \frac{1}{x^2} Q(x) y = 0$,

where R, P, Q: analytic at $x_0 = 0$, $R(0) = 1$.

It follows that $R(x) = 1, \quad P(x) = 0, \quad Q(x) = x$

<u>Indicial equation</u> : $f(s) = s(s-1) + \underset{0}{P_0} s + \underset{0}{Q_0} = 0 \Rightarrow f(s) = s(s-1) = 0 \Rightarrow \boxed{s = 0, 1}$

$\cdot \quad s_1 = 1, \quad s_2 = 0.$

BEWARE!

(3.) Replace $y(x) = x^{s_1} \sum_{k=0}^{\infty} A_k x^k, \qquad \boxed{A_0 \neq 0.}$

Recursive function $g_n(s) = R_n (s-n)(s-n-1) + P_n (s-n) + Q_n, \quad n \geq 1.$

It follows that $\underline{g_n(s) \equiv 0}$ for $\underline{n \neq 1}$, $\quad g_1(s) = Q_1 = 1$

Recurrence formula for A_k:

$$f(\overset{s}{\underset{1}{s_1}} + k)\, A_k = -\sum_{n=0}^{k} g_n(s_1+k)\, A_{k-n}, \qquad \underline{k \geq 1}.$$

$$\Rightarrow \quad f(s_1+k)\, A_k = -g_1(s_1+k) \cdot A_{k-1}, \qquad k = 1, 2, 3, \ldots, \qquad A_0 \neq 0,$$

$$\Rightarrow \quad k(1+k)\, A_k = -A_{k-1}$$

$\underline{k=1}$: $\quad 1 \cdot 2\, \cancel{A_1} = -A_0$

$\underline{k=2}$: $\quad 2 \cdot 3\, \cancel{A_2} = -\cancel{A_1}$

$\underline{k=3}$: $\quad 3 \cdot 4\, A_3 = -A_2$

$\vdots$

$\underline{k=k}$: $\quad k(k+1) A_k = -A_{k-1}$

$$\xrightarrow{\text{Multiply}} (1 \cdot 2 \cdot 3 \cdots k)^2 (k+1)\, A_k = (-1)^k A_0$$

$$\Rightarrow \quad A_k = \frac{(-1)^k}{(k!)^2 (k+1)} A_0$$

Hence, $$y(x) = A_0 x \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 (k+1)} x^k \equiv A_0\, u_1(x), \qquad u_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 (k+1)} x^k$$

(4.) Since $s_1 - s_2 = 1 - 0 = 1$: integer > 0, we have to <u>check</u> the recurrence formula for $s = s_2 = 0$ and $\underline{k=1}$:

$$\underline{k=1}: \quad \underbrace{f(\overset{s}{\underset{0}{s_2}} + \underset{1}{k})}_{0} \cdot \underset{1}{A_k} = -\overbrace{g_1(s_2+k)}^{1} \cdot \underset{0}{A_{k-1}}$$

$$\Rightarrow \quad 0 \cdot A_1 = -A_0 \neq 0, \quad \text{which is } \underline{\text{impossible}}.$$

Hence, it is not possible to find a second independent solution by this method.

(5.) The second independent solution is of the form

$$y_2(x) = C\,u_1(x)\ln x + \sum_{k=0}^{\infty} B_k\, x^{k+\overset{=0}{s_2}}, \qquad C \neq 0,$$

where B_k are functions of C.

The general solution is then

$$y(x) = \underbrace{A_0\,u_1(x)}_{\text{from (3) above}} + y_2(x), \qquad C, A_0: \text{arbitrary}.$$

$\boxed{\text{OPTIONAL}}$

(6.) Let $\quad y_2(x) = C\,u_1(x)\ln x + \sum_{k=0}^{\infty} B_k\, x^k. \qquad (1)$

This $u_1(x)$ satisfies the ODE: $\quad u_1''(x) + \frac{1}{x}\,u_1(x) = 0.$

From Eq. (1),

$$y_2'(x) = C\,u_1'(x)\ln x + \frac{C}{x}\,u_1(x) + \sum_{k=0}^{\infty} k B_k\, x^{k-1}$$

$$y_2''(x) = C\,u_1''(x)\ln x + \frac{2C}{x}\,u_1'(x) - \frac{C}{x^2}\,u_1(x) + \sum_{k=0}^{\infty} k(k-1) B_k\, x^{k-2}$$

$y_2(x)$ has to satisfy the ODE: (in canonical form for convenience): (we put it)

$$y_2''(x) + \frac{1}{x}\,y_2(x) = 0 \implies \left[C\,u_1''(x)\ln x + \frac{2C}{x}\,u_1'(x) - \frac{C}{x^2}\,u_1(x) + \sum_{k=0}^{\infty} k(k-1) B_k\, x^{k-2}\right]$$

$$+ \frac{C}{x}\,u_1(x)\ln x + \overbrace{\sum_{k=0}^{\infty} B_k\, x^{k-1}}^{\text{let } k-1 \to k-2} = 0$$

$$\iff C\underbrace{\left[u_1''(x) + \frac{1}{x}\,u_1(x)\right]}_{0}\ln x + \frac{2C}{x}\,u_1'(x) - \frac{C}{x^2}\,u_1(x) + \sum_{k=0}^{\infty} B_{k-1} x^{k-2} + \sum_{k=0}^{\infty} (k-1)k\,B_k\, x^{k-2} = 0$$

$$\iff \frac{2C}{x}\,u_1'(x) - \frac{C}{x^2}\,u_1(x) + \sum_{k=1}^{\infty}\left[B_{k-1} + k(k-1)\,B_k\right] x^{k-2} = 0. \qquad (2)$$

From part (3), above,

$$u_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2\,(k+1)}\, x^{k+1}$$

$$\rightarrow \frac{1}{x^2}\,u_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2\,(k+1)}\, x^{k-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{[(k-1)!]^2\,k}\, x^{k-2}$$

and $$u_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} x^k$$

$$\to \frac{1}{x} u_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} x^{k-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{[(k-1)!]^2} x^{k-2}$$

Eq. (2) is written as

$$\sum_{k=1}^{\infty} \left\{ C \frac{(-1)^{k-1} (2k-1)}{[(k-1)!]^2 k} + B_{k-1} + k(k-1)B_k \right\} x^{k-2} = 0 \quad , \text{ all } x.$$

It follows that

$$C \cdot (-1)^{k-1} \frac{2k-1}{[(k-1)!]^2 k} + B_{k-1} + k(k-1) B_k = 0, \qquad k=1,2,3,\ldots$$

This is the recurrence formula for the unknown B_k's.

$\underline{k=1}$: $\quad C + B_0 + 0 \cdot B_1 = 0 \Rightarrow \underline{B_0 = -C}, \quad \underline{B_1 : \text{arbitrary}}$. $\underline{\text{Set}}$ $\boxed{B_1 \equiv 0}$

$\underline{k=2}$: $\quad -\frac{3}{2} C + \cancel{B_1} + 2 \cdot 1 B_2 = 0 \Rightarrow \underline{B_2 = \frac{3}{4} C}$

$\underline{k=3}$: $\quad \frac{5}{4 \cdot 3} C + B_2 + 3 \cdot 2\, B_3 = 0 \Rightarrow B_3 = -\frac{1}{6} B_2 - \frac{5}{12} C = -\frac{1}{8} C - \frac{5}{12} C = \underline{-\frac{13}{12} C}$.

etc ...

In this way, we find that all B_k's $(k \neq 1)$ are proportional to C.

(IV) ODE: $\quad x^2 y'' + xy' + (x^2 - p^2) y = 0 \quad , \quad p \geq 0.$

(1.) This is the <u>Bessel equation</u>. The general solution is

$$y(x) = \begin{cases} c_1 \cdot J_p(x) + c_2 J_{-p}(x), & p \neq \text{integer} \\ c_1 \cdot J_p(x) + c_2 Y_p(x), & p = \text{integer} \end{cases} \equiv Z_p(x)$$

c_1, c_2 : arbitrary

(2.) <u>Suppose</u> that <u>p=integer</u> (Repeat the solution for $p \neq$ integer !)

$$\left.\begin{aligned} y(x) &= c_1 J_p(x) + c_2 Y_p(x) &\to\ & y(1) = c_1 J_p(1) + c_2 Y_p(1) = A \\ y'(x) &= c_1 J_p'(x) + c_2 Y_p'(x) &\to\ & y'(1) = c_1 J_p'(1) + c_2 Y_p'(1) = B \end{aligned}\right\}$$

system with two unknowns, c_1, c_2, where A, B: <u>known</u>.

We solve this system of linear equations by any method (it's simple!)

$$c_1 = \frac{\begin{vmatrix} A & Y_p(1) \\ B & Y_p'(1) \end{vmatrix}}{\begin{vmatrix} J_p(1) & Y_p(1) \\ J_p'(1) & Y_p'(1) \end{vmatrix}} = \frac{A\,Y_p'(1) - B\,Y_p(1)}{J_p(1)\,Y_p'(1) - Y_p(1)\,J_p'(1)},$$

$$c_2 = \frac{\begin{vmatrix} J_p(1) & A \\ J_p'(1) & B \end{vmatrix}}{\begin{vmatrix} J_p(1) & Y_p(1) \\ J_p'(1) & Y_p'(1) \end{vmatrix}} = \frac{B\,J_p(1) - A\,J_p'(1)}{J_p(1)\,Y_p'(1) - Y_p(1)\,J_p'(1)}.$$

This solution exists if $J_p(1)\,Y_p'(1) - Y_p(1)\,J_p'(1) \neq 0$, and we then find c_1 and c_2 in terms of A, B.

Note: This is an example where ~~an~~ 2nd-order ODE is solved with 2 conditions at one point; it's called an initial-value problem.

(3.) Take $p=0$; then the ODE becomes

$$x^2y'' + xy' + x^2y = 0,$$

with solution

$$y(x) = c_1\,J_0(x) + c_2\,Y_0(x).$$

Note that $J_0(x)$ is smooth at $x=0$, while $Y_0(x)$ blows up logarithmically at $x=0$.

If we require that $y(0) =$ finite, then we must set $c_2 \equiv 0$.

Hence, $y(x) = c_1 J_0(x) \rightarrow A = y(0) = c_1 \underbrace{J_0(0)}_{1} = c_1 \rightarrow \boxed{c_1 = A}$

Solution: $y(x) = A\,J_0(x)$ (unique solution, with 1 condition!)

(V.) ODE: $x^2y'' + xy' - \left(x^2 + \frac{1}{4}\right)y = 0$

Let $x = iX$. Then $y(x) = Y(X)$ and $\frac{dy}{dx} = \frac{d}{d(iX)}\,Y(X) = -i\,\frac{d}{dX}\,Y(X)$

$$\rightarrow x\frac{dy}{dx} = X\;Y'(X).$$

Similarly,

$$x^2\frac{d^2y}{dx^2} = X^2\,\frac{d^2Y}{dX^2},\quad \text{while} \quad x^2 = -X^2.$$

So, the ODE for $Y(X)$ is $\quad X^2 Y'' + X Y' + (X^2 - \underbrace{\tfrac{1}{4}}_{p^2}) Y = 0,$

which is the Bessel equation for $p = 1/2$. Hence,

$$Y(X) = c_1 J_{1/2}(X) + c_2 J_{-1/2}(X) = c_1 \sqrt{\frac{2}{\pi X}} \sin X + c_2 \sqrt{\frac{2}{\pi X}} \cos X$$

$$\rightarrow \quad y(x) = c_1 J_{1/2}(+ix) + c_2 J_{-1/2}(ix) = c_1 \sqrt{\frac{2}{\pi i x}} \sin(ix) + c_2 \sqrt{\frac{2}{\pi i x}} \cos(ix)$$

$$= \bar{c}_1 \sqrt{\frac{2}{\pi x}} \sinh x + \bar{c}_2 \sqrt{\frac{2}{\pi x}} \cosh x,$$

where $\quad \bar{c}_1 = i c_1 \frac{1}{\sqrt{i}}, \quad \bar{c}_2 = \frac{1}{\sqrt{i}} c_2, \quad \sin(ix) = i \sinh x, \quad \cos(ix) = \cosh x.$