# Properties of Laurent Series

$$\sum_{n=-\infty}^{\infty} C_n(z-z_0)^n \equiv G(z)$$

(i) If series converges, it has to converge in $\rho_1 < |z-z_0| < \rho_2$
(ii) If series converges, $G(z)$ is unique.
(iii) If series converges, $G(z)$ is continuous and analytic.
(iv) If series converges, the series can be integrated and differentiated term by term. (they converge in the same region)

## Singularities

singularity of $f(z)$: a point where $f(z)$ is not analytic:
- branch points: $f(z)$ is not single valued
- other

ex $f'(z) = Ln = \left(\frac{1+z}{1-z}\right) = \underbrace{Ln(1+z)}_{0 \text{ at } z=-1} - \underbrace{Ln(1-z)}_{0 \text{ at } z=1}$ (Ln has branch point at $z=0$)

Other singularities, $z_0$:

$$f(z) = \sum_{n=-\infty}^{\infty} C_n(z-z_0)^n \quad \text{(definition)}$$

(i) $z_0$ is called removable singularity if $\lim_{z\to z_0} f(z)$: finite and $f(z)$: analytic in the neighborhood of $z_0$. $(z \neq z_0)$

$$f(z) = \cdots + \frac{C_{-m}}{(z-z_0)^m} + \frac{C_{-m+1}}{(z-z_0)^{m-1}} + \cdots + C_0 + C_1(z-z_0) + \cdots$$

$\lim_{z\to z_0} f(z)$: finite $\equiv A \rightarrow C_{-M}=0,\ -M<0$

$\rightarrow f(z) = C_0 + C_1(z-z_0) + \cdots + C_n(z-z_0)^n$ ; Taylor series

$\rightarrow \boxed{C_0 = A}$ Can always define $f(z_0) \equiv A = C_0$.

ex $f(z) = \frac{\sin z}{z}$, $z_0 = 0$ $\lim_{z\to 0} \frac{\sin z}{z} = 1 = A$

$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \cdots$ $\therefore z_0 = 0$: removable discontinuity.

(ii) $C_k = 0$ for $K < -M$, $M > 0$, $C_M \neq 0$

$$f(z) = \frac{C_{-M}}{(z-z_0)^M} + \frac{C_{-M+1}}{(z-z_0)^{M-1}} + \dots + \frac{C_{-1}}{(z-z_0)} + C_0 + C_1(z-z_0) + \dots$$

by definition, $z_0$ is $M^{th}$ order pole of $f(z)$

$M=1$: simple pole $M=2$: double pole $M=3$: triple pole.

$$z_0 : M^{th} \text{ order pole} \longleftrightarrow \begin{cases} (z-z_0)^M f(z) : \text{analytic} \\ \underbrace{\lim_{z \to z_0} \left[(z-z_0)^M f(z)\right]}_{C_{-M}} : \text{finite} \neq 0 \end{cases}$$

(iii) $f(z) = \dots \frac{C_{-M}}{(z-z_0)^M} + \dots$ infinite number of negative powers of $z - z_0$

$\to z_0$: essential singularity

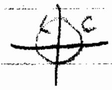
<u>Definition:</u> $C_{-1}$: residue of $f(z)$

$$f(z) = \dots + \frac{C_{-M}}{(z-z_0)^M} + \dots + \frac{C_{-1}}{(z-z_0)} + C_0 + C_1(z-z_0) + \dots$$

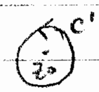
$$\oint_C f(z)dz = \dots + \cancelto{0}{\int \frac{C_{-M}}{(z-z_0)^M} dz} + \dots + C_{-1}\int_C \frac{dz}{z-z_0} + C_0 \cancelto{0}{\oint_C dz} + \dots$$

$$\downarrow$$

$$\boxed{\oint_C f(z)dz = C_{-1} 2\pi i}$$ always valid.

<u>Recall:</u> 


$$\oint_C z^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$




$$\oint_{C'} (z-z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

<u>ex</u> $f(z) = \frac{P_N(z)}{Q_M(z)}$ $P_N, Q_M$: polynomials of degree $N, M$.

possible isolated singularities: $Q_M(z_i) = 0$

removable or poles