

PROFESSOR: Hi, everyone. So for the first part of this course, we've learned basically the ins and outs of solving linear systems of equations. Today we're going to do a little review of the basic concepts. Hopefully we'll see a few of them in the following problem.

We're given a square matrix A -- a three by three square matrix A -- where the last entry is a parameter k . And this parameter will vary. And we'll see what happens to the system of equations Ax equal to $[2, 3, 7]$, for which k it has a unique solution, for which k it has infinitely many solutions. Then we'll find the LU decomposition. And finally, we'll write down the complete solution, the system.

So I'll give you a few minutes to work this problem on your own. And then please come back and see how I do it.

All right, welcome back. So let's start with part A. For which k does this system, Ax equal $[2, 3, 7]$, have a unique solution? So what do we know about square systems of linear equations? They have a unique solution when the matrix A is invertible.

So now, when is A invertible? It is invertible when it is of full rank. And how do we figure this out? We do it by performing row operations. We do it by doing eliminations on the matrix.

But since we want to simulate an exam setting, it always pays off to see what tasks lie ahead of us. So in part C, we're asked to find the LU decomposition. This means that when we do row operations, we'd better keep track what row operations we're doing exactly. In particular, we'll write them down as elementary matrices. And in part D, we'll be asked to compute the complete solution. And therefore it's good to do row operations on the augmented matrix A . So let's do this.

I'm going to write this. The augmented matrix A is the following beast, $1, 1, 1; 1, 2, 3; 3, 4, k;$ and then $2, 3, 7$. So first thing, we subtract a multiple of row one from row two. And it's exactly negative 1 times the first row plus the second.

Let me write down the corresponding elementary matrix that does this. It's $E_{(2,1)}$. And it's lower diagonal with 1's on the diagonal. And it's going to be exactly minus 1 in the first entry of the second row. So we get $1, 1, 1, 2; 0, 1, 2, 1$. And we copy down the third row.

Now we subtract a multiple of the first row from the third one. And let me write this here. Yeah,

we'll multiply the first row by negative 3, and add it to the third one. This is accomplished by the elementary matrix $E_{(3,1)}$ which is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$. OK, $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & k-3 & 7-3 \cdot 2 \end{pmatrix}$. We copy the first two rows. And then the third one will be $0, 1, k-3, 7-6$.

We have essentially one last row operation to perform. Let me do it here. So we'll subtract the second row from the third one. And we'll get $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-5 & 1 \end{pmatrix}$. And this was achieved by the elementary matrix $E_{(3,2)}$, which was $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. Because we multiplied the second row by negative 1 and added it to the third one.

So we got to a matrix, which is upper triangular. And we want to figure out: for which value of the parameter k is this matrix of full rank. This is a pivot. This is a pivot. And we want this one to be a pivot as well. And that happens when $k-5$ is not 0. So when k is different from 5, the matrix A is of full rank. And therefore the system $A \cdot x$ equals to $[2, 3, 7]$ has a unique solution.

Now part B. For which k do we have infinitely many solutions? So when are we in such a situation? We are in such a situation when the null space of the matrix A is nontrivial. So the null space will be nontrivial when this $k-5$ number here, which is what's the pivot in the first case, is 0. So $k-5$ equals to 0.

You see there's a little caveat here. When k is equal to 5, we get the third row of the augmented matrix, $0, 0, 0$, equal to 0. This means that the matrix is actually consistent, and we indeed have a solution. But if this entry were nonzero, then we would get no solutions.

Now off to part C. We want to compute the LU decomposition. Well, we already got what the matrix U is through performing row operations. It's this guy here. Let me write it down-- $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & k-5 \end{pmatrix}$. And when k is equal to 4, this entry's negative 1.

And now what about the matrix L ? Well, how did we get to this U ? We had the matrix A . We got the matrix U . And what we did was first we applied $E_{(2,1)}$. Then we applied $E_{(3,1)}$ and then $E_{(3,2)}$.

We get A by inverting this equation. So it's going to be $E_{(2,1)}^{-1} E_{(3,1)}^{-1} E_{(3,2)}^{-1} U$. And this is our matrix L . And we know it's fairly easy to invert these elementary matrices E . We flip the signs of the off-diagonal entries.

I'm not going to write down again the inverses of these guys. I'm just going to write the product of the inverses. And that's also very easy to compute. Because one, we invert the signs. We just send the numbers in their corresponding entry of L . I mean the following thing. So minus 1 becomes a 1. And it comes here, in its respective entry.

For $E_{(3,1)}$, we flip the sign, first 3, and we plug it in here. And for $E_{(3,2)}$, we flip the sign of this guy, becomes 1, and we plug it in here. So give me a few moments to erase the board, and then I'll do part four.

We're back, and we're going to do part D now. So we need to find the complete solution of the system for all k . And we saw that for k equal to 5, the system had many, many solutions. And for k not equal to 5, it had only one.

So first let's look at the case k not equal to 5, when the matrix A was invertible. It's not hard to see what the solution of the system then is. It's going to be-- let me just write it down. So when k is not equal to 5, this was nonzero. Therefore, x_3 needs to be 0. When x_3 is 0, we have x_2 plus 2 times x_3 . So x_2 plus 0 equals 1. Therefore x_2 is 1. And then we go back to the first row. We have x_1 plus x_2 plus x_3 equals 2. So x_1 plus 1 plus 0 equals 2. So x_1 plus 1 equals 2. And therefore x_1 is 1. Good.

Now what about k equal to 5? Then we see that x_3 is a free variable. So the solution will be $[x_1, x_2, x_3]$. x_3 can be any number c . From the second row, we'll get the value of x_2 . It's 1 minus 2 times x_3 . So 1 minus 2 times c . And x_1 is 2 minus x_2 minus x_3 . So let me rewrite this. It's 2 minus x_2 is 1 minus-- aah, chalk-- $2c$, minus c , 1 minus $2c$, c .

So we'll decompose this vector here into a component which is independent of c and a component which is a multiple of c . So this is 2 minus 1. It should be $[1, 1, 0]$ plus c times-- we'll have two c minus c , c , so c times 1. Here we'll have minus $2c$ so negative 2. And here we'll have 1. And thus we get the particular solution for the system and the special solution for the system.

We're kind of done here. If you're at an exam, you should immediately start the next problem. Good luck, and I'll see you later.