PROFESSOR: Hi, everyone. Welcome back. So today, I'd like to talk about positive definite matrices. And specifically, we're going to analyze several properties of positive definite matrices. And specifically, we're going to look at why each one of these following statements is true.

So first off, why every positive definite matrix is invertible. Why the only positive definite projection matrix is the identity matrix. If d is a diagonal matrix with positive entries, show that it must also be positive definite. And then lastly, if S is a symmetric matrix where the determinant S is bigger than 0, show why this might not necessarily imply that it's positive definite.

So I'll let you think about these for a moment. And I'll come back in a second. Hi, everyone. Welcome back. OK. So let's take a look at part A. So part A is asking why every positive definite matrix is invertible.

Well, let's just recall that if A is a matrix and if A is invertible, then this necessarily implies that the determinant of A is non-zero. And I'm going to just write out detA as the product of the eigenvalues of A. So lambda 1 to lambda n are the eigenvalues of A.

OK. In addition, if A is positive definite, what does this say about the eigenvalues of A? Well, it says that each eigenvalue of A, lambda 1, lambda 2, dot, dot, dot, to lambda n, each one of them must be bigger than 0. So this statement that each eigenvalue of A is bigger than 0 is completely equivalent to A being positive definite for symmetric matrices A.

So if I have a whole bunch of eigenvalues and each of them are bigger than 0, what does this say about detA? Well, I can take the product of all these eigenvalues. And of course, the product of a whole bunch of positive numbers must also be positive. So if the quantity is positive, then it certainly cannot equal 0. So this proves that detA is not equal to 0. Hence, A must be invertible.

OK. So for part B, we're asked to show that the only positive definite projection

matrix is the identity matrix. So again, how do we tackle this problem? We're going to look at the eigenvalues.

So remember if P is a projection, what does it say about the eigenvalues of P? Well, it says that the eigenvalues of P are either 0 or 1. So this is point one. Point two, if P is a positive definite, what does that say about the eigenvalues of P? Well, as I've noted before, it means that the eigenvalues are bigger than 0.

So if P is a projection and it's positive definite, the only possible eigenvalues that are both 0 and 1 and bigger than 0 are 1. So the conclusion is that the eigenvalues of P must all equal 1.

So which matrix has eigenvalues 1 and is also symmetric? Well, the only matrix that satisfies this property is the identity matrix. Now you might ask, how do you actually show that? Well, you could argue as follows. If P is diagonalizable and every symmetric matrix is diagonalizable-- so I'm not making this up.

So if P is diagonalizable, then you can always write P as some matrix, U, times a diagonal matrix. And we know in this case, the diagonal matrix has eigenvalues 1. So it's actually the identity matrix times the inverse of the eigenvector matrix. But of course, this is just U times U inverse, which then gives me the identity at the end.

So U times the identity times U inverse. This is just U times U inverse. And of course, U and U inverse collapse back down to the identity. So this shows you that the only matrix that has eigenvalues of 1 is the identity matrix. So that's just to cross all the T's and dot all the I's.

OK. For part C, we're given D as a diagonal matrix with positive entries on the diagonal. Now we have to show that it's positive definite. OK. So let me write D as follows. I'll just write it like this. I'm going to use a compact notation, which is sometimes seen diag d1, d2, dn. So d is a diagonal matrix whose elements along the diagonal are d1, d2, dot, dot, dot, to dn. Now what does it mean for a matrix to be positive definite?

Well, it means that for any x, for any vector x, I have to look at the product x

transpose Dx. And I have to show that it's bigger than 0. And I should qualify this and say that the vector we're looking at is x not equal to 0. So we're now looking at the 0 vector.

But for D to be positive definite, we have to show that x transpose Dx is bigger than 0. This is just one way to show that it's positive definite. It's not the only way. So if I write x out using components, x1, x2, dot, dot, dot, to xn-- I'll write it like this-- then you can work out the quantity x transpose Dx. And we see that we get a sum of squares. We get d1 times x1 squared plus d2 x2 squared plus dot, dot, dot plus dn xn squared.

Now by definition, each coefficient is positive. A sum of a square is positive. so A product of a positive number with a square is positive. And then of course, a sum of positive numbers is going to be positive. So this means the whole thing is positive.

Now there's other more efficient ways of getting at this using other tricks we know. For example, if we're given a diagonal matrix, we know its eigenvalues are already d1, d2, dot, dot, dot, to dn. And we know that a matrix with positive eigenvalues is already positive definite. But this is kind of starting from the base to show that it's positive definite. And now lastly, let's look at part D.

So S is a symmetric matrix with detS bigger than 0. Show that S might not necessarily be positive definite. So there's lots of counter examples. I only need to construct one. So I'm looking at S, which is a symmetric matrix. So I'm just going to throw in some numbers on some off diagonals. So I'll just pick one.

And now I need to pick some numbers along the diagonal. And the easiest way to do it is just to pick two negative numbers on the diagonal. Because if there's a negative number on a diagonal, then we know S can't be positive definite. And I'll say more about that in a second.

So I can just pick negative 2 and negative 3. So let's quickly check what detS is. Well, it's negative 2 times negative 3 minus 1. So this gives me 6 minus 1, which is 5. So by construction, detS is positive, which is good.

And then as I mentioned before, if there's negative elements along the diagonal of the matrix, that matrix can't be positive definite. Well, why is that? Well, suppose I wanted to take a look at this upper left component, negative 3, and it's negative, how do I show that that implies S is not positive definite. Well, what I can do is I can look at the product x transpose Sx. And what I do is I look at it. I just take one value of x.

So we know that this has to be positive for every value of x. So I can pick any value I want. So I can take x, say, 1, 0 transpose. And when I do this, we end up getting that x transpose Sx is equal to negative 3.

So notice how by taking 1 in the first entry and 0 on the second entry, that picks out the upper left corner negative 3. If I were to take 0 here and 1 here, it would pick out the negative 2 entry and S. So by picking 1 in entry i of a vector x, and then computing this product x transpose Sx, I pick out the i-th element along the diagonal. And since that element is negative, this shows me that along the direction 1, 0, the product x transpose Sx is also negative. And hence, S can't possibly be positive definite.

OK. So just to summarize, we've taken a look at a couple matrices and a couple different properties of positive definite matrices. And notably, we've used the eigenvalues to get a handle of the positive definite matrices. And I hope these provide some useful tricks. And I'll see you next time.