PROFESSOR: Hi, and welcome. Today, we're going to do a problem about powers of a matrix. Our problem is first to find a formula for the k-th power of this matrix C. This is a two by two matrix that depends on variables a and b.

And the second part of our problem is to calculate C to the 100th in the special case where a and b are -1. You can hit pause now, and I'll give you a minute to do the problem yourself. And then, I'll come back and we can do it together.

OK. We're back. Now what's the first step in finding powers of a matrix? Well, we need to find the eigenvalues and eigenvectors of this matrix. So how do we do that? We compute the determinant of C minus lambda I. This is the determinant of this matrix. 2b minus a minus lambda, 2b minus 2a, a minus b, and 2a minus b minus lambda.

OK. If you compute this, well, we have a lambda squared term. OK. Our lambda term, if you look at it, you'll see we get 2b minus a plus 2a minus b, which is just a plus b. And we have a negative sign. And it's negative a plus b times lambda. And our last term is a little tougher to compute. So I'll let you do it yourself. But you're just going to get plus ab. And this will factor as lambda minus a times lambda minus b.

So our eigenvalues are just a and b. Now we need to find our eigenvectors. So how do we do that? Well, what we need to do is we need to look at C minus a times the identity. And we need to find the null space of this matrix.

So what do we get here? We get 2b minus 2a. And then our next entry here, we get 2a minus b minus a. So this is a minus b. Good. So you can see that this matrix has the same columns and the same rows. And so you can see that a vector in the null space, since this column is -2 times this column, we can see that our first eigenvector is just-- or 1, 2, I should say. It's just 1, 2.

Good. Well, I guess we have space to do the second one too. Why not? So let's write out the second one also. Here, we're subtracting b instead of a. You get b

minus a. You get 2b minus 2a. We get a minus b. And what do we have here? We have 2a minus 2b.

So now, what's in the null space of this matrix? Well, what you can see is that this column is -1 times that column. So our second eigenvector is just going to be 1, 1.

And I should remind you that if you have a harder example, you can just find these null spaces by elimination like we always do. Great. Now we have our eigenvalues and our eigenvectors. So now we can write C in a nice easy way that allows us to take powers of it.

So what's that way? So that's C equals S lambda S inverse. So this is just, what is S? Remember, S is our matrix of eigenvectors. So S is the matrix 1, 2, 1, 1.

Good. Now what is lambda? Lambda is the matrix of eigenvalues. Right? So it's just a and b. Those are the diagonal entries of my lambda matrix. And then, we just find S inverse. So we just take negative signs here and recall that we have to divide by the determinant. And the determinant of this matrix is just -1.

So we just change the signs there. Good. So this is our nice decomposition of C. Now how do we take powers of C? Well, C to the k is just S lambda to the k, S inverse. 1, 1, 2, 1. a to the k. b to the k. -1, 1, 2, -1.

Good. And multiplying these matrices together, just do a little arithmetic here. Got a bunch of powers of a and b. Because we take powers of the eigenvalues. We have here, we have 2b to the k minus a to the k. Have a to the k minus b to the k. 2b to the k minus 2a to the k.

And finally, we get 2a to the k minus b to the k. And this is our k-th power matrix. Good. A quick check. It's always good to check your work here. Let's plug in k equals 1. And what do we get? We get 2b minus a, a minus b, 2b minus 2a, and 2a minus b. And if we can go all the way back to our matrix at the very beginning, all the way back here, that agrees perfectly with what we started with.

So that's good. That means that we did this decomposition right. Good. So now,

we've computed the k-th power of this matrix. Let's do a particular example. So let's plug in a and b are -1. So a equals b equals -1. And k equals 100. Then what do we get?

Well -1 to the 100th is just 1. So we're just plugging in 1 for b to the k and a to the k everywhere. And we just get, in this case, C to the 100th is just 1, 0, 0, 1. It's just the identity matrix.

Great. Great. OK. Now to summarize, how do we take powers of a matrix? Well, first we diagonalize. We write our matrix as S lambda, S inverse. And then, we just take powers of the diagonal matrix.