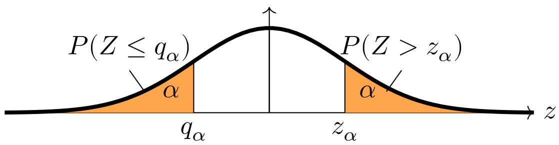
Class 22 in-class problems, 18.05, Spring 2022

Concept questions

Concept question 1. Critical values



1. $z_{0.025} =$

(a) -1.96 (b) -0.95 (c) 0.95 (d) 1.96 (e) 2.87

2. $-z_{0.16} =$

(a) -1.33 (b) -0.99 (c) 0.99 (d) 1.33 (e) 3.52

1. **Solution:** $z_{0.025} = 1.96$. By definition $P(Z > z_{0.025}) = 0.025$. This is the same as $P(Z \le z_{0.025}) = 0.975$. Either from memory, a table or using the R function `qnorm(0.975)` we get the result.

2.**Solution:** $-z_{0.16} = -0.99$. We recall that $P(|Z| < 1) \approx 0.68$. Since half the leftover probability is in the right tail we have $P(Z > 1) \approx 0.16$. Thus $z_{0.16} \approx 1$, so $-z_{0.16} \approx -1$.

Board questions

Problem 1. Computing confidence intervals

The data 4, 1, 2, 3 is drawn from $N(\mu, \sigma^2)$ with μ unknown.

(a) *Find a 90% z confidence interval for μ, given that $\sigma = 2$.*

For the remaining parts, suppose σ is unknown.

(b) *Find a 90% t confidence interval for μ.*

(c) *Find a 90% χ^2 confidence interval for σ^2.*

(d) *Find a 90% χ^2 confidence interval for σ.*

(e) *Given a normal sample with $n = 100$, $\overline{x} = 12$, and $s = 5$, find the rule-of-thumb 95% confidence interval for μ.*

Solution: $\overline{x} = 2.5$, $s^2 = 1.667$, $s = 1.29$, $\sigma/\sqrt{n} = 1$, $s/\sqrt{n} = 0.645$.

(a) $z_{0.05} \approx 1.645$: 90% z confidence interval for μ is

$$\left[\overline{x} - z_{0.05} \cdot \frac{\sigma}{\sqrt{n}},\ \overline{x} + z_{0.05} \cdot \frac{\sigma}{\sqrt{n}}\right] \approx [0.856, 4.144] = 2.5 \pm 1.645.$$

(b) $t_{0.05} \approx 2.353$ (3 degrees of freedom): 90% t confidence interval for μ is

$$\left[\overline{x} - t_{0.05} \cdot \frac{s}{\sqrt{n}},\ \overline{x} + t_{0.05} \cdot \frac{s}{\sqrt{n}}\right] \approx [0.981,\ 4.019] = 2.5 \pm 1.519$$

(c) $c_{0.05} \approx 7.815$, $c_{0.95} \approx 0.352$ (3 degrees of freedom): 90% χ^2 confidence interval for σ^2 is

$$\left[\frac{(n-1)s^2}{c_{0.05}},\ \frac{(n-1)s^2}{c_{0.95}}\right] \approx [0.640,\ 14.211].$$

(d) Take the square root of the interval in 3. $[0.780,\ 3.770]$.

(e) The rule of thumb is written for z, but with $n = 100$ the $t(99)$ and standard normal distributions are very close, so we can assume that $t_{0.025} \approx 2$. Thus the 95% confidence interval is $12 \pm 2 \cdot 5/10 = [11,\ 13]$.

Problem 2. Confidence intervals and non-rejection regions

Suppose $x_1, \ldots, x_n \sim N(\mu, \sigma^2)$ with σ known.

Consider two intervals:

1. The z confidence interval around $\overline{x}$ at confidence level $1 - \alpha$.

2. The z non-rejection region for $H_0 : \mu = \mu_0$ at significance level α.

Compute and sketch these intervals to show that:

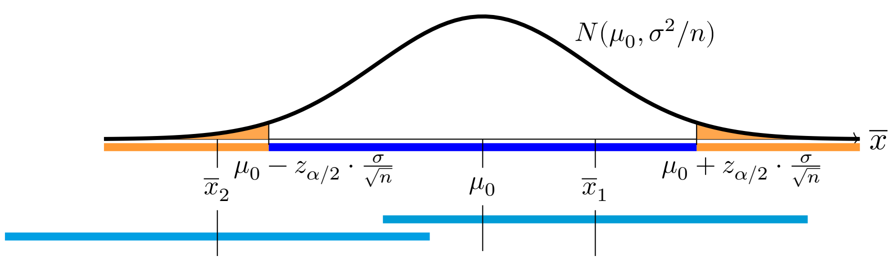
$$\mu_0 \text{ is in the first interval} \quad \Leftrightarrow \quad \overline{x} \text{ is in the second interval.}$$

Solution:

Confidence interval: $\overline{x} \pm z_{\alpha/2} \cdot \dfrac{\sigma}{\sqrt{n}}$

Non-rejection region: $\mu_0 \pm z_{\alpha/2} \cdot \dfrac{\sigma}{\sqrt{n}}$

Since the intervals are the same width they either both contain the other's center or neither one does.



Problem 3. Polling

For a poll to find the proportion θ of people supporting X we know that a $(1-\alpha)$ confidence interval for θ is given by

$$\left[\bar{x} - \frac{z_{\alpha/2}}{2\sqrt{n}},\ \bar{x} + \frac{z_{\alpha/2}}{2\sqrt{n}}\right].$$

***(a)** How many people would you have to poll to have a margin of error of 0.01 with 95% confidence? (You can do this in your head.)*

***(b)** How many people would you have to poll to have a margin of error of 0.01 with 80% confidence. (You'll want R or other calculator here.)*

***(c)** If $n = 900$, compute the 95% and 80% confidence intervals for θ.*

Solution: **(a)** Need $1/\sqrt{n} = 0.01$ So $n = 10000$.

(b) $\alpha = 0.2$, so $z_{\alpha/2} =$ `qnorm(0.9)` $= 1.2816$. So we need $\dfrac{z_{\alpha/2}}{2\sqrt{n}} = 0.01$. This gives $n = 4106$.

(c) 95% interval: $\overline{x} \pm \dfrac{1}{\sqrt{n}} = \overline{x} \pm \dfrac{1}{30} = \overline{x} \pm 0.0333$

80% interval: $\overline{x} \pm z_{0.1} \cdot \dfrac{1}{2\sqrt{n}} = \overline{x} \pm 1.2816 \cdot \dfrac{1}{60} = \overline{x} \pm 0.021$.

Discussion questions

1. Width of confidence intervals

The quantities n, $c =$ confidence, $\overline{x}$, σ all appear in the z confidence interval for the mean.

How does the width of a confidence interval for the mean change if:

1. We increase n and leave the others unchanged?

2. We increase c and leave the others unchanged?

3. We increase μ and leave the others unchanged?

4. We increase σ and leave the others unchanged?

(A) it gets wider (B) it gets narrower (C) it stays the same.

Solution: 1. Narrower. More data decreases the variance of $\bar{x}$

2. Wider. Greater confidence requires a bigger interval.

3. No change. Changing μ will tend to shift the location of the intervals.

4. Wider. Increasing σ will increase the uncertainty about μ.