Class 17 in-class problems, 18.05, Spring 2022

Concept questions

Concept question 1. What would a frequentist say?
Each day Jaam arrives X hours late to class, with $X \sim$ uniform$(0, \theta)$, where θ is unknown. Jon models his initial belief about θ by a prior pdf $f(\theta)$. After Jaam arrives x hours late to the next class, Jon computes the likelihood function $\phi(x|\theta)$ and the posterior pdf $f(\theta|x)$.

Which of these probability computations would the frequentist consider valid?

1. none	*5. prior and posterior*
2. prior	*6. prior and likelihood*
3. likelihood	*7. likelihood and posterior*
4. posterior	*8. prior, likelihood and posterior.*

Solution: 3. likelihood

Both the prior and posterior are probability distributions on the possible values of the unknown parameter θ, i.e. a distribution on hypothetical values. The frequentist does not consider them valid.

The likelihood $\phi(x|\theta)$ is perfectly acceptable to the frequentist. It represents the probability of data from a repeatable experiment, i.e. measuring how late Jane is each day. Conditioning on θ is fine. This just fixes a model parameter θ. It doesn't require computing probabilities of values of θ.

Concept question 2. Is it a statistic. *Suppose $x_1, \ldots, x_n$ is a sample from $N(\mu, \sigma^2)$, where μ and σ are unknown.*

Is each of the following a statistic?

(a) *The median of $x_1, \ldots, x_n$.*

(b) *The interval from the 0.25 quantile to the 0.75 quantile of $N(\mu, \sigma^2)$.*

(c) *The standardized mean $\dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$.*

(d) *The set of sample values less than 1 unit from $\bar{x}$.*

(e) *The $z = \dfrac{\overline{x} - 5}{3/\sqrt{n}}$.*

(f) $z = \dfrac{\overline{x} - \mu_0}{\sigma_0/\sqrt{n}}$, *where μ_0 and σ_0 are given values,*

(a) Yes. The median only depends on the data $x_1, \ldots, x_n$.

(b) No. This interval depends only on the distribution parameters μ and σ. It does not consider the data at all.

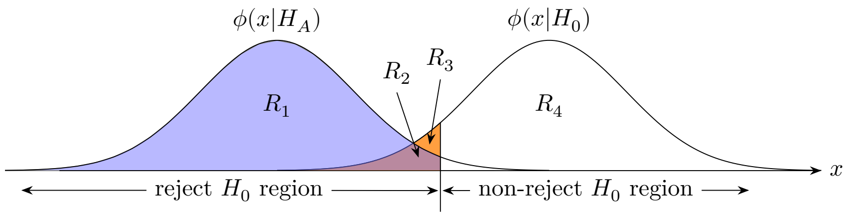
(c) No. this depends on the values of the unknown parameters μ and σ.

(d) Yes. $\bar{x}$ depends only on the data, so the set of values within 1 of $\bar{x}$ can all be found by working with the data.

(e) Yes. This is computed from the data and other known values.

(f) Yes. This is computed from the data and other known values.

Concept question 2. Picture the significance. *The null and alternate pdfs are shown on the following plot*



The significance level of the test is given by the area of which region?

1. R_1
2. R_3
3. $R_1 + R_2$
4. $R_2 + R_3 + R_4$
5. R_2
6. R_4
7. $R_2 + R_3$
8. None of these

Solution: 7. $R_2 + R_3$. This is the area under the pdf for H_0 above the rejection region.

Board questions

Problem 1. Testing coins

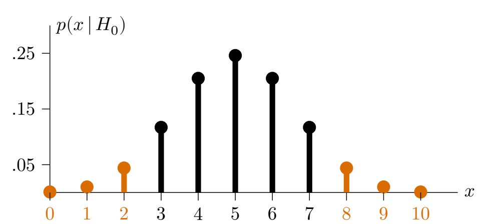
Suppose we have a coin with an unknown probability of heads θ.

Test statistic x = number of heads in 10 tosses.

Null hypothesis H_0: $\theta = 0.5$ (fair coin).

Alternative hypothesis H_A: $\theta \neq 0.5$ (unfair coin, two-sided).

(a) *The rejection region is are the values of x shown in orange. What's the significance level?*



x	0	1	2	3	4	5	6	7	8	9	10
$p(x\|H_0)$	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001

(b) *For significance level $\alpha = 0.05$, find a two-sided rejection region.*

Solution: **(a)** $\alpha = 0.11$

x	0	1	2	3	4	5	6	7	8	9	10
$p(x\|H_0)$	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001

(b) $\alpha = 0.05$

x	0	1	2	3	4	5	6	7	8	9	10
$p(x\|H_0)$	0.001	0.010	0.044	0.117	0.205	0.246	0.205	0.117	0.044	0.010	0.001

Problem 2. z statistic

Suppose we know the following about our null hypothesis significance test.

- H_0: *data follows a* $N(5, 10^2)$
- H_A: *data follows a* $N(\mu, 10^2)$ *where* $\mu \neq 5$.
- *Test statistic:* z = *standardized* $\overline{x}$.
- *Data: 64 data points with* $\overline{x} = 6.25$.
- *Significance level set to* $\alpha = 0.05$.

(a) *Find the rejection region; draw a picture.*

(b) *Find the z-value; add it to your picture.*

(c) *Decide whether or not to reject* H_0 *in favor of* H_A.

(d) *Find the p-value for this data; add to your picture.*

(e) *What's the connection between the answers to (b), (c) and (d)?*

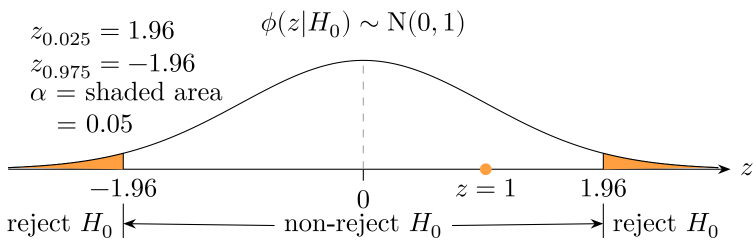
The null distribution $\phi(z \,|\, H_0) \sim N(0, 1)$

(a) The rejection region is $|z| > 1.96$, i.e. 1.96 or more standard deviations from the mean.

(b) Standardizing $z = \dfrac{\overline{x} - 5}{5/4} = \dfrac{1.25}{1.25} = 1$.

(c) Do not reject since z is not in the rejection region.

(d) Use a two-sided p-value $p = P(|Z| > 1) = 0.32$.



(e) The z-value not being in the rejection region tells us exactly the same thing as the p-value being greater than the significance, i.e. don't reject the null hypothesis H_0.

Problem 2. More coins

Two coins: probability of heads is 0.5 for C_1; *and 0.6 for* C_2.

We pick one at random, flip it 8 times and get 6 heads.

Here are the probability tables for the two coins

k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$p(k\|\theta = 0.5)$	*0.004*	*0.031*	*0.109*	*0.219*	*0.273*	*0.219*	*0.109*	*0.031*	*0.004*
$p(k\|\theta = 0.6)$	*0.001*	*0.008*	*0.041*	*0.124*	*0.232*	*0.279*	*0.209*	*0.090*	*0.017*

(a) H_0 = *'The coin is* C_1*'* H_A = *'The coin is* C_2*'*

Do you reject H_0 *at the significance level* $\alpha = 0.05$*?*

(Hint: First decide if this test is two-sided, left-sided or right-sided. Then determine the rejection region.)

(b) H_0 = *'The coin is* C_2*'* H_A = *'The coin is* C_1*'*

Do you reject H_0 *at the significance level* $\alpha = 0.05$*?*

(c) *Do your answers to (a) and (b) seem paradoxical*

Solution: (a) Since $0.6 > 0.5$ we use a right-sided rejection region.

Under H_0 the probability of heads is 0.5. Using the table we find a one sided rejection region $\{7, 8\}$. That is we will reject H_0 in favor of H_A only if we get 7 or 8 heads in 8 tosses.

Since the value of our data $x = 6$ is not in our rejection region we do not reject H_0.

(b) Since $0.6 > 0.5$ we use a left-sided rejection region.

Now under H_0 the probability of heads is 0.6. Using the table we find a one sided rejection region $\{0, 1, 2\}$. That is we will reject H_0 in favor of H_A only if we get 0, 1 or 2 heads in 8 tosses.

Since the value of our data $x = 6$ is not in our rejection region we do not reject H_0.

(c) The fact that we don't reject C_1 in favor of C_2 or C_2 in favor of C_1 reflects the asymmetry in NHST. The null hypothesis is the cautious choice. That is, we only reject H_0 if the data is extremely unlikely when we assume H_0. This is not the case for either C_1 or C_2.