Class 15 in-class problems, 18.05, Spring 2022

Concept questions

Concept question 1. More Beta

Suppose your prior $f(\theta)$ in the bent coin example is Beta$(6, 8)$. You flip the coin 7 times, getting 2 heads and 5 tails. What is the posterior pdf $f(\theta|x)$?

(a) Beta(2,5)

(b) Beta(11,10)

(c) Beta(6,8)

(d) Beta(8,13)

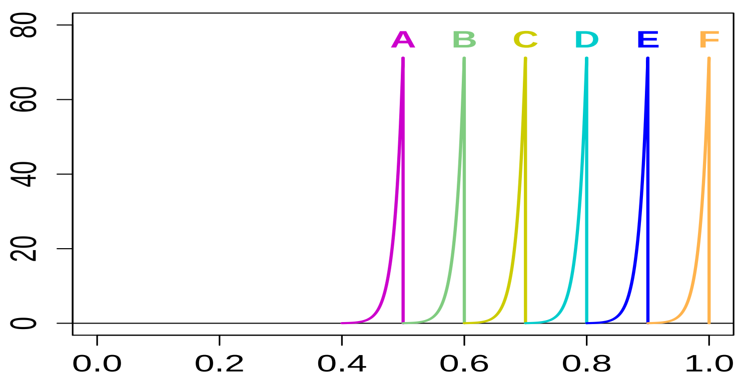
Concept question 2. Strong priors

Say we have a bent coin with unknown probability of heads θ.

We are convinced that $\theta \leq 0.7$.
Our prior is uniform on $[0, 0.7]$ and 0 from 0.7 to 1.

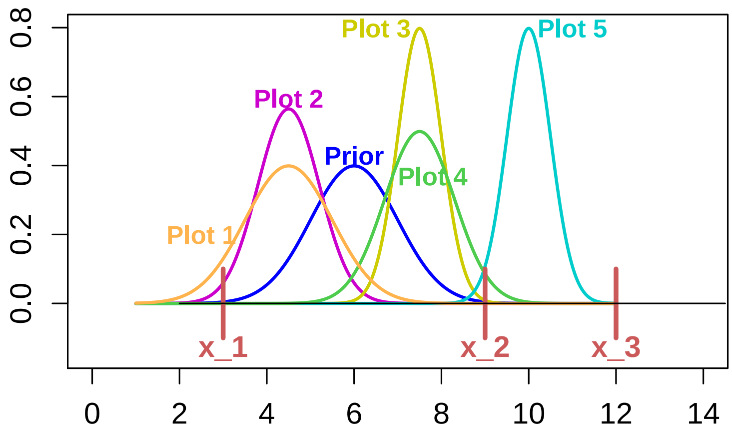
We flip the coin 65 times and get 60 heads.

Which of the graphs below is the posterior pdf for θ?



Concept question 3. Normal priors, normal likelihood

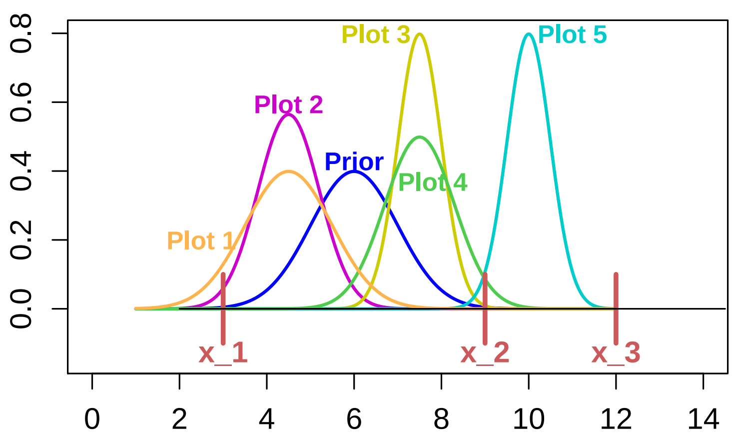
(a)



Blue graph = prior,   Red lines = data in order: 3, 9, 12

Which plot is the posterior to just the first data value?

Concept question 4. Normal priors, normal likelihood
(b)

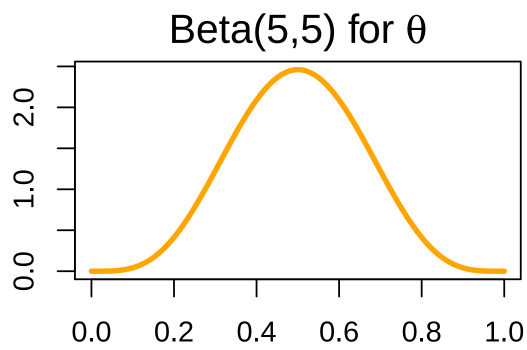


Blue graph = prior,   Red lines = data in order: 3, 9, 12

Which graph is posterior to all 3 data values?

Board questions

Problem 1. Beta priors
Suppose you are testing a new medical treatment with unknown probability of success θ. You don't know θ, but your prior belief is that it's probably not too far from 0.5. You capture this intuition with a Beta(5,5) prior on θ.



To sharpen this distribution you take data and update the prior.

- Beta(a, b): $f(\theta) = \dfrac{(a+b-1)!}{(a-1)!(b-1)!}\theta^{a-1}(1-\theta)^{b-1}$
- Treatment has prior $f(\theta) \sim \text{Beta}(5, 5)$

(a) Suppose you test it on 25 patients and have 20 successes.
– Find the posterior distribution on θ.
– Identify the type of the posterior distribution.

(b) Suppose you recorded the order of the results and got

$$S\,S\,S\,S\,F\,S\,S\,S\,S\,S\,F\,F\,S\,S\,S\,F\,S\,F\,S\,S\,S\,S\,S\,S\,S$$

(20 S and 5 F). Find the posterior based on this data.

(c) Using your answer to (b) give an integral for the posterior predictive probability of success with the next patient.

(d) Don't do in class. Use what you know about pdf's to evaluate the integral without computing it directly

Problem 2. Normal-normal updating
For data $x_1, \dots, x_n$ with data mean $\bar{x} = \frac{x_1 + \ldots + x_n}{n}$

$$a = \frac{1}{\sigma^2_{\text{prior}}}, \quad b = \frac{n}{\sigma^2}, \quad \mu_{\text{post}} = \frac{a\mu_{\text{prior}} + b\bar{x}}{a+b}, \quad \sigma^2_{\text{post}} = \frac{1}{a+b}.$$

Suppose we have one data point $x = 2$ drawn from $\text{N}(\theta, 3^2)$

Suppose θ is our parameter of interest with prior $\theta \sim \text{N}(4, 2^2)$.

(a) Identify μ_{prior}, σ_{prior}, σ, n, and $\bar{x}$.

(b) Make a Bayesian update table, but leave the posterior as an unsimplified product.

(c) Use the updating formulas to find the posterior.

Problem 2. Normal/normal
Question. On a basketball team the average career free throw percentage over all players follows a $\text{N}(75, 6^2)$ distribution. In a given year individual players free throw percentage is $\text{N}(\theta, 4^2)$ where θ is their career average.

This season Sophie Lie made 85 percent of her free throws. What is the posterior expected value of her career percentage θ?

In class examples and discussion

1. Likelihood principle

Suppose the prior has been set. Let x_1 and x_2 be two sets of data. Which of the following are true?

(a) If the likelihoods $\phi(x_1|\theta)$ and $\phi(x_2|\theta)$ are the same then they result in the same posterior.

(b) If the likelihoods $\phi(x_1|\theta)$ and $\phi(x_2|\theta)$ are proportional (as functions of θ) then they result in the same posterior.

(c) If two likelihood functions are proportional then they are equal.

Extra problems

Extra 1. Conjugate priors

Which of the following likelihood/prior pairs are conjugate?

	hypothesis	data	prior	likelihood
(a) Exponential/Normal	$\theta \in [0, \infty)$	x	$\mathrm{N}(\mu_{\text{prior}}, \sigma^2_{\text{prior}})$	$\exp(\theta)$
	θ	x	$c_1 \exp\left(-\frac{(\theta-\mu_{\text{prior}})^2}{2\sigma^2_{\text{prior}}}\right)$	$\theta \mathrm{e}^{-\theta x}$
(b) Exponential/Gamma	$\theta \in [0, \infty)$	x	$\mathrm{Gamma}(a, b)$	$\exp(\theta)$
	θ	x	$c_1 \theta^{a-1} \mathrm{e}^{-b\theta}$	$\theta \mathrm{e}^{-\theta x}$
(c) Binomial/Normal	$\theta \in [0, 1]$	x	$\mathrm{N}(\mu_{\text{prior}}, \sigma^2_{\text{prior}})$	$\mathrm{binomial}(N, \theta)$
(fixed N)	θ	x	$c_1 \exp\left(-\frac{(\theta-\mu_{\text{prior}})^2}{2\sigma^2_{\text{prior}}}\right)$	$c_2\, \theta^x (1-\theta)^{N-x}$

1. none	2. a	3. b	4. c
5. a,b	6. a,c	7. b,c	8. a,b,c