Expectation, Variance and Standard Deviation for Continuous Random Variables
Class 6, 18.05
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1 Learning Goals

1. Be able to compute and interpret expectation, variance, and standard deviation for continuous random variables.
2. Be able to compute and interpret quantiles for discrete and continuous random variables.

2 Introduction

So far we have looked at expected value, standard deviation, and variance for discrete random variables. These summary statistics have the same meaning for continuous random variables:

- The expected value $\mu = E[X]$ is a measure of location or central tendency.
- The standard deviation σ is a measure of the spread or scale.
- The variance $\sigma^2 = \text{Var}(X)$ is the square of the standard deviation.

To move from discrete to continuous, we will simply replace the sums in the formulas by integrals. We will do this carefully and go through many examples in the following sections. In the last section, we will introduce another type of summary statistic, quantiles. You may already be familiar with the 0.5 quantile of a distribution, otherwise known as the median or 50^{th} percentile.

3 Expected value of a continuous random variable

Definition: Let X be a continuous random variable with range $[a, b]$ and probability density function $f(x)$. The expected value of X is defined by

$$E[X] = \int_a^b x f(x)\, dx.$$

Let's see how this compares with the formula for a discrete random variable:

$$E[X] = \sum_{i=1}^{n} x_i p(x_i).$$

The discrete formula says to take a weighted sum of the values x_i of X, where the weights are the probabilities $p(x_i)$. Recall that $f(x)$ is a probability **density**. Its units are prob/(unit of X). So $f(x)\,dx$ represents the probability that X is in an infinitesimal range of width dx around x. Thus we can interpret the formula for $E[X]$ as a weighted integral of the values x of X, where the weights are the probabilities $f(x)\,dx$.

As before, the expected value is also called the mean or average.

3.1 Examples

Let's go through several example computations. Where the solution requires an integration technique, we push the computation of the integral to the appendix.

Example 1. Let $X \sim \text{uniform}(0,1)$. Find $E[X]$.

Solution: X has range $[0,1]$ and density $f(x) = 1$. Therefore,

$$E[X] = \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \boxed{\frac{1}{2}}.$$

Not surprisingly the mean is at the midpoint of the range.

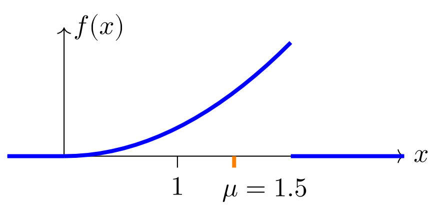
Example 2. Let X have range $[0,2]$ and density $\frac{3}{8}x^2$. Find $E[X]$.

Solution:

$$E[X] = \int_0^2 x f(x)\,dx = \int_0^2 \frac{3}{8}x^3\,dx = \left.\frac{3x^4}{32}\right|_0^2 = \boxed{\frac{3}{2}}.$$

Does it make sense that this X has mean is in the right half of its range?

Solution: Yes. Since the probability density increases as x increases over the range, the average value of x should be in the right half of the range.

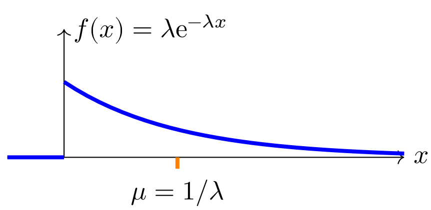


μ is "pulled" to the right of the midpoint 1 because there is more mass to the right.

Example 3. Let $X \sim \exp(\lambda)$. Find $E[X]$.

Solution: The range of X is $[0,\infty)$ and its pdf is $f(x) = \lambda e^{-\lambda x}$. So (details in appendix)

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \left. -xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right|_0^\infty = \boxed{\frac{1}{\lambda}}.$$

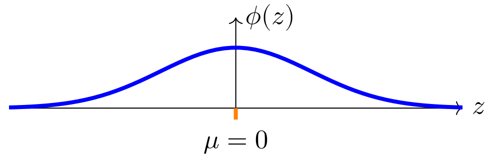


Mean of an exponential random variable

Example 4. Let $Z \sim \mathrm{N}(0,1)$. Find $E[Z]$.

Solution: The range of Z is $(-\infty, \infty)$ and its pdf is $\phi(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}$. So (details in appendix)

$$E[Z] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z\mathrm{e}^{-z^2/2}\,dz = -\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\Big|_{-\infty}^{\infty} = \boxed{0}.$$



The standard normal distribution is symmetric and has mean 0.

3.2 Properties of $E[X]$

The properties of $E[X]$ for continuous random variables are the same as for discrete ones:

1. If X and Y are random variables on a sample space Ω then

$$E[X+Y] = E[X] + E[Y]. \qquad \text{(linearity I)}$$

2. If a and b are constants then

$$E[aX+b] = aE[X] + b. \qquad \text{(linearity II)}$$

Example 5. In this example we verify that for $X \sim \mathrm{N}(\mu, \sigma)$ we have $E[X] = \mu$.

Solution: Example (4) showed that for standard normal Z, $E[Z] = 0$. We could mimic the calculation there to show that $E[X] = \mu$. Instead we will use the linearity properties of $E[X]$. In the class 5 notes on manipulating random variables we showed that if $X \sim \mathrm{N}(\mu, \sigma^2)$ is a normal random variable we can **standardize** it:

$$Z = \frac{X - \mu}{\sigma} \sim \mathrm{N}(0,1).$$

Inverting this formula we have $X = \sigma\, Z + \mu$. The linearity of expected value now gives

$$E[X] = E[\sigma\, Z + \mu] = \sigma\, E[Z] + \mu = \mu$$

3.3 Expectation of Functions of X

This works exactly the same as the discrete case. if $h(x)$ is a function then $Y = h(X)$ is a random variable and

$$E[Y] = E[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx.$$

Example 6. Let $X \sim \exp(\lambda)$. Find $E[X^2]$.

Solution: Using integration by parts we have

$$E[X^2] = \int_0^{\infty} x^2 \lambda \mathrm{e}^{-\lambda x}\,dx = \left[-x^2\mathrm{e}^{-\lambda x} - \frac{2x}{\lambda}\mathrm{e}^{-\lambda x} - \frac{2}{\lambda^2}\mathrm{e}^{-\lambda x}\right]_0^{\infty} = \boxed{\frac{2}{\lambda^2}}.$$

4 Variance

Now that we've defined expectation for continuous random variables, the definition of variance is identical to that of discrete random variables.

Definition: Let X be a continuous random variable with mean μ. The variance of X is

$$\text{Var}(X) = E[(X - \mu)^2].$$

4.1 Properties of Variance

These are exactly the same as in the discrete case.

1. If X and Y are independent then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

2. For constants a and b, $\text{Var}(aX + b) = a^2\text{Var}(X)$.

3. **Theorem**: $\text{Var}(X) = E[X^2] - E[X]^2 = E[X^2] - \mu^2$.

For Property 1, note carefully the requirement that X and Y are independent.

Property 3 gives a formula for $\text{Var}(X)$ that is often easier to use in hand calculations. The proofs of properties 2 and 3 are essentially identical to those in the discrete case. We will not give them here.

Example 7. Let $X \sim \text{uniform}(0, 1)$. Find $\text{Var}(X)$ and σ_X.

Solution: In Example 1 we found $\mu = 1/2$. Next we compute

$$\text{Var}(X) = E[(X - \mu)^2] = \int_0^1 (x - 1/2)^2 \, dx = \boxed{\frac{1}{12}}.$$

Example 8. Let $X \sim \exp(\lambda)$. Find $\text{Var}(X)$ and σ_X.

Solution: In Examples 3 and 6 we computed

$$E[X] = \int_0^\infty x\lambda e^{-\lambda x} \, dx = \frac{1}{\lambda} \quad \text{and} \quad E[X^2] = \int_0^\infty x^2\lambda e^{-\lambda x} \, dx = \frac{2}{\lambda^2}.$$

So by Property 3,

$$\text{Var}(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \quad \text{and} \quad \sigma_X = \frac{1}{\lambda}.$$

We could have skipped Property 3 and computed this directly from $\text{Var}(X) = \int_0^\infty (x - 1/\lambda)^2 \lambda e^{-\lambda x} \, dx$.

Example 9. Let $Z \sim \text{N}(0, 1)$. Show $\text{Var}(Z) = 1$.

Note: The notation for normal variables is $X \sim \text{N}(\mu, \sigma^2)$. This is certainly suggestive, but as mathematicians we need to prove that $E[X] = \mu$ and $\text{Var}(X) = \sigma^2$. Above we showed $E[X] = \mu$. This example shows that $\text{Var}(Z) = 1$, just as the notation suggests. In the next example we'll show $\text{Var}(X) = \sigma^2$.

Solution: Since $E[Z] = 0$, we have

$$\text{Var}(Z) = E[Z^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} \, dz.$$

(using integration by parts with $u = z$, $v' = z\mathrm{e}^{-z^2/2} \;\Rightarrow\; u' = 1$, $v = -\mathrm{e}^{-z^2/2}$)

$$= \frac{1}{\sqrt{2\pi}}\left(-z\mathrm{e}^{-z^2/2}\Big|_{-\infty}^{\infty}\right) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-z^2/2}\,dz.$$

The first term equals 0 because the exponential goes to zero much faster than z grows at both $\pm\infty$. The second term equals 1 because it is exactly the total probability integral of the pdf $\varphi(z)$ for $\mathrm{N}(0,1)$. So $\mathrm{Var}(X) = 1$.

Example 10. Let $X \sim \mathrm{N}(\mu, \sigma^2)$. Show $\mathrm{Var}(X) = \sigma^2$.

Solution: This is an exercise in change of variables. Letting $z = (x-\mu)/\sigma$, we have

$$\begin{aligned}\mathrm{Var}(X) = E[(X-\mu)^2] &= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty}(x-\mu)^2\mathrm{e}^{-(x-\mu)^2/2\sigma^2}\,dx \\ &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2\mathrm{e}^{-z^2/2}\,dz = \sigma^2.\end{aligned}$$

The integral in the last line is the same one we computed for $\mathrm{Var}(Z)$.

5 Quantiles

Definition: The median of X is the value x for which $P(X \le x) = 0.5$, i.e. the value of x such that $P(X \le x) = P(X \ge x)$. In other words, X has equal probability of being above or below the median, and each probability is therefore $1/2$. In terms of the cdf $F(x) = P(X \le x)$, we can equivalently define the median as the value x satisfying $F(x) = 0.5$.

Think: What is the median of Z?

Solution: By symmetry, the median is 0.

Example 11. Find the median of $X \sim \exp(\lambda)$.

Solution: The cdf of X is $F(x) = 1 - \mathrm{e}^{-\lambda x}$. So the median is the value of x for which $F(x) = 1 - \mathrm{e}^{-\lambda x} = 0.5$.. Solving for x we find: $\boxed{x = (\ln 2)/\lambda}$.

Think: In this case the median does not equal the mean of $\mu = 1/\lambda$. Based on the graph of the pdf of X can you argue why the median is to the left of the mean.

Definition: The p^{th} quantile of X is the value q_p such that $P(X \le q_p) = p$.

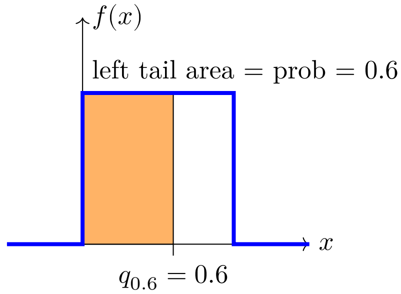
Notes. 1. In this notation the median is $q_{0.5}$.

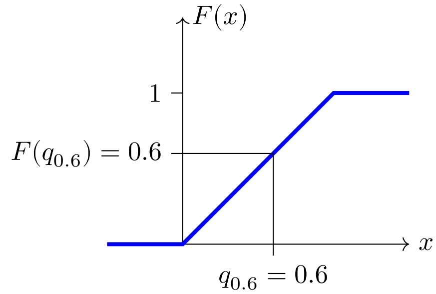
2. We will usually write this in terms of the cdf: $F(q_p) = p$.

With respect to the pdf $f(x)$, the quantile q_p is the value such that there is an area of p to the left of q_p and an area of $1-p$ to the right of q_p. In the examples below, note how we can represent the quantile graphically using either area of the pdf or height of the cdf.

Example 12. Find the 0.6 quantile for $X \sim U(0,1)$.

Solution: The cdf for X is $F(x) = x$ on the range [0,1]. So $q_{0.6} = 0.6$.



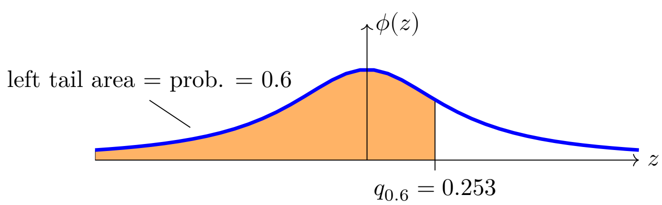


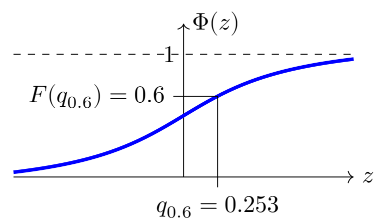
$$q_{0.6}\text{: left tail area} = 0.6 \;\Leftrightarrow\; F(q_{0.6}) = 0.6$$

Example 13. Find the 0.6 quantile of the standard normal distribution.

Solution: We don't have a formula for the cdf, so we use the R 'quantile function' `qnorm`.

$$q_{0.6} = \texttt{qnorm(0.6, 0, 1)} = 0.25335$$





$$q_{0.6}\text{: left tail area} = 0.6 \;\Leftrightarrow\; F(q_{0.6}) = 0.6$$

Quantiles give a useful measure of **location** for a random variable. We will use them more in coming lectures.

5.1 Percentiles, deciles, quartiles

For convenience, quantiles are often described in terms of percentiles, deciles or quartiles. The 60^{th} percentile is the same as the 0.6 quantile. For example you are in the 60^{th} percentile for height if you are taller than 60 percent of the population, i.e. the **probability** that you are taller than a randomly chosen person is 60 percent.

Likewise, deciles represent steps of 1/10. The third decile is the 0.3 quantile. Quartiles are in steps of 1/4. The third quartile is the 0.75 quantile and the 75^{th} percentile.

6 Appendix: Integral Computation Details

From Example 3 Let $X \sim \exp(\lambda)$. Find $E[X]$.

The range of X is $[0, \infty)$ and its pdf is $f(x) = \lambda e^{-\lambda x}$. Therefore

$$E[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty \lambda x e^{-\lambda x}\,dx$$

(using integration by parts with $u = x$, $v' = \lambda e^{-\lambda x} \Rightarrow u' = 1$, $v = -e^{-\lambda x}$)

$$= -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$
$$= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}.$$

We used the fact that $xe^{-\lambda x}$ and $e^{-\lambda x}$ go to 0 as $x \to \infty$.

From Example 4 Let $Z \sim \mathrm{N}(0, 1)$. Find $E[Z]$.

The range of Z is $(-\infty, \infty)$ and its pdf is $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. By symmetry the mean must be 0. The only mathematically tricky part is to show that the integral converges, i.e. that the mean exists at all (some random variable do not have means, but we will not encounter this very often.) For completeness we include the argument, though this is not something we will ask you to do. We first compute the integral from 0 to ∞:

$$\int_0^\infty z\phi(z)\,dz = \frac{1}{\sqrt{2\pi}}\int_0^\infty ze^{-z^2/2}\,dz.$$

The u-substitution $u = z^2/2$ gives $du = z\,dz$. So the integral becomes

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty ze^{-z^2/2}\,dz. \;=\; \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-u}\,du \;=\; -e^{-u}\big|_0^\infty \;=\; 1$$

Similarly, $\int_{-\infty}^0 z\phi(z)\,dz = -1$. Adding the two pieces together gives $E[Z] = 0$.

From Example 6 Let $X \sim \exp(\lambda)$. Find $E[X^2]$.

$$E[X^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx$$

(using integration by parts with $u = x^2$, $v' = \lambda e^{-\lambda x} \Rightarrow u' = 2x$, $v = -e^{-\lambda x}$)

$$= -x^2 e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2xe^{-\lambda x}\,dx$$

(the first term is 0, for the second term use integration by parts: $u = 2x$, $v' = e^{-\lambda x} \Rightarrow u' = 2$, $v = -\frac{e^{-\lambda x}}{\lambda}$)

$$= -2x\frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty + \int_0^\infty \frac{e^{-\lambda x}}{\lambda}\,dx$$
$$= 0 - 2\frac{e^{-\lambda x}}{\lambda^2}\Big|_0^\infty = \frac{2}{\lambda^2}.$$