The Existence and Uniqueness Theorem for Linear Systems

For simplicity, we stick with $n = 2$, but the results here are true for all n. There are two questions about the following general linear system that we need to consider.

$$\begin{aligned} x' &= a(t)x + b(t)y \\ y' &= c(t)x + d(t)y \end{aligned} \quad ; \text{ in matrix form, } \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

The first is from the previous section: to show that all solutions are of the form

$$\mathbf{x} = c_1\mathbf{x}_1 + x_2\mathbf{x}_2,$$

where the $\mathbf{x}_i$ form a fundamental set, that is, no $\mathbf{x}_i$ is a constant multiple of the other). (The fact that we can write down *all* solutions to a linear system in this way is one of the main reasons why such systems are so important.)

An even more basic question for the system (1) is: how do we know that it *has* two linearly independent solutions? For systems with a constant coefficient matrix A, we showed in the previous chapters how to solve them explicitly to get two independent solutions. But the general non-constant linear system (1) does not have solutions given by explicit formulas or procedures.

The answers to these questions are based on following theorem.

Theorem 2 Existence and uniqueness theorem for linear systems.

If the entries of the square matrix $A(t)$ are continuous on an open interval I containing t_0, then the initial value problem

$$\boldsymbol{x}' = A(t)\,\boldsymbol{x}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{2}$$

has one and only one solution $\boldsymbol{x}(t)$ on the interval I.

The proof is difficult and we shall not attempt it. More important is to see how it is used. The following three theorems answer the questions posed for the 2×2 system (1). They are true for $n > 2$ as well, and the proofs are analogous.

In the following theorems, *we assume the entries of $A(t)$ are continuous on an open interval I.* Here the conclusions are valid on the interval I, for example, I could be the whole t-axis.

Theorem 2A Linear independence theorem.

Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two solutions to (1) on the interval I, such that at some point t_0 in I, the vectors $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$ are linearly independent. Then

a) the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent on I, and

b) the vectors $\mathbf{x}_1(t_1)$ and $\mathbf{x}_2(t_1)$ are linearly independent at every point t_1 of I.

Proof. a) By contradiction. If they were dependent on I, one would be a constant multiple of the other, say $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$. Then $\mathbf{x}_2(t_0) = c_1\mathbf{x}_1(t_0)$, showing them dependent at t_0. □

b) By contradiction. If there were a point t_1 on I where they were dependent, say $\mathbf{x}_2(t_1) = c_1\mathbf{x}_1(t_1)$, then $\mathbf{x}_2(t)$ and $c_1\mathbf{x}_1(t)$ would be solutions to (1) which agreed at t_1. Hence, by the uniqueness statement in Theorem 2, $\mathbf{x}_2(t) = c_1\mathbf{x}_1(t)$ on all of I, showing them linearly dependent on I. □

Theorem 2B General solution theorem.

a) The system (1) has two linearly independent solutions.

b) If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any two linearly independent solutions, then every solution $\mathbf{x}$ can be written in the form (3), for some choice of c_1 and c_2:

$$\mathbf{x} \;=\; c_1\mathbf{x}_1 + c_2\mathbf{x}_2. \tag{3}$$

Proof. Choose a point $t = t_0$ in the interval I.

a) According to Theorem 2, there are two solutions $\mathbf{x}_1$, $\mathbf{x}_2$ to (1), satisfying respectively the initial conditions

$$\mathbf{x}_1(t_0) \;=\; \mathbf{i}, \qquad \mathbf{x}_2(t_0) \;=\; \mathbf{j}\,, \tag{4}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the usual unit vectors in the xy-plane. Since the two solutions are linearly independent when $t = t_0$, they are linearly independent on I, by Theorem 5.2A.

b) Let $\mathbf{u}(t)$ be a solution to (1) on I. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent at t_0 by Theorem 2, using the parallelogram law of addition we can find constants c_1' and c_2' such that

$$\mathbf{u}(t_0) \;=\; c_1'\mathbf{x}_1(t_0) + c_2'\mathbf{x}_2(t_0). \tag{5}$$

The vector equation (5) shows that the solutions $\mathbf{u}(t)$ and $c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t)$ agree at t_0. Therefore by the uniqueness statement in Theorem 2, they are equal on all of I; that is,

$$\mathbf{u}(t) \;=\; c_1'\mathbf{x}_1(t) + c_2'\mathbf{x}_2(t) \qquad \text{on } I.$$

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