Phase Lines Applet

1. Guided Tour of the Applet

Use the link on the session page to open the *Phase Lines* applet. We will configure the applet to display the logistic equation.

- On the lower left the autonomous equation is displayed in yellow. Select the equation $y' = (1 - y)y - a$.
- Set the slider for a (on the lower right) to 0.
- Find the *Phase Line* checkbox and make sure it is checked. Find the *Bifurcation Diagram* checkbox and make sure it is unchecked. Click the *Clear* button.

The main graphing window displays the direction field and any equilibrium solutions. Stable equilibria are indicated in green, unstable in red, and semistable in cyan. You should see one green line at $y = 1$ and one red line at $y = 0$

Click in the main graphing window to draw some solution curves. Notice that when the cursor is placed over the graphing window a yellow vector parallel with the vector field is displayed, and the values of t, y, and y′ appear in yellow below the graphing window. You can clear the solution curves by pressing the *Clear* key at the left underneath the main graphing window.

At this point the main graph should contain a direction field and some integral curves that look like the ones we derived for the logistic equation earlier in this session. Notice that the solution curves around the stable equilibrium all head towards it and the solution curves around the unstable equilibrium move away from it.

Click both the *Clear* button under the main window and the one under the narrow phase line window. This will bring the screen back to its pristine state.

The phase line window shows a the phase line, i.e. the vertical y-axis. The critical points are indicated by green and red dots. Notice that they are at exactly the same heights as the corresponding lines in the main window. If there were a semistable critical point it would be shown in cyan.

You should see three yellow arrows on the phase line. The top arrow points down; this corresponds to the downward slope of the direction field when $y > 1$. This indicates all solution curves in this region are decreasing. Likewise the middle arrow points up and the bottom arrow points down, corresponding to increasing and decreasing solutions respectively.

The top and middle arrows point towards the top (green) dot. Green lines and dots indicate a *stable equilibrium* and the arrows confirm that any solution that starts near this equilibrium will go asymptotically towards it. Similarly, the middle and bottom arrows point away from the bottom (red) dot. Red indicates an *unstable equilibrium* and the arrows confirm that any solution that starts near this equilibrium will move away from it.

Each of the menu choices includes a parameter a which can be set using the slider at lower right.

Now check the *Bifurcation Diagram* checkbox to reveal the *bifurcation diagram*. Notice that the horizontal axis is for the paramater a. The diagram displays the *locus* of critical points as they depend on a. The critical points are colored green, red, or cyan as in the other plots. The phase line for the selected value of a is also displayed and moves as the value of a is changed. It always matches the current phase line shown in the phase line window.

2. Example: Oryx Redux

The Kenyan government wants to establish a game preserve on which it will allow the hunting of oryx. It wants to sell licenses to kill 250 oryx each year and wants to know how big a preserve to establish to guarantee that this is sustainable. It's known that with no hunting the population (measured in kilo-oryx) follows the logistic equation

$$y' = (a - y)y.$$

Here, a is the stable population of oryx in the absence of hunting. It is proportional to the area of the preserve. With our 1/4 kilo-oryx per year hunt, we then have

$$y' = f(y) \;=\; (a - y)y - 0.25 \;=\; -y^2 + ay - 0.25.$$

Choose this equation in the applet. (Be careful, there is another equation that looks a lot like this one.) Make sure the *Phase Line* and *Bifurcation Diagram* are showing.

Start with a small and slowly increase it.

Before reading any further, use the applet to help answer the following question, "What is the smallest value of a that will allow for a sustainable population?"

When a is small there are no critical points and the arrows on the phase line point down. The population crashes no matter what its initial value.

When $a = 1$, a single *semistable* equilibrium appears. You can see the graph of $f(y)$ at the lower left of the applet; it's a parabola, opening downward. As you increase a the parabola rises until at $a = 1$ it has a single root.

We could also see this using the quadratic formula to solve for the critical points (roots of $f(y)$):

$$y = a/2 + -(1/2)(\sqrt{(}a^2 - 1))$$

When $|a| < 1$ there are no roots, when $|a| = 1$ there is exactly one root $y = a/2 = 1/2$.

So the Kenyan Chamber of Commerce recommends a preserve of size, $a = 1$ kilo-oryz. It reasons that if the population starts above the critical value $1/2$ the model says it will go asymptotically to $1/2$, where it will stay forever.

A group called 'Friend of the Oryx' claims this is not wise. They argue that if the population falls every so slightly below the critical population of $1/2$ then it crashes.

When you increase a further, things get better. There are two critical points. The farther they are from each other, the more stable the population system is. Friends of the Oryx argues for a preserve of size $a = 1.25$ kilo-oryx. This has a stable equilibrium at at $y = 1$ and an unstable equilibrium at $y = .25$, which leaves a large safety buffer. The oryx population would have to dip .75 kilo-oryx below its stable equilibrium before it could crash.

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