The Simple Harmonic Oscillator

Let's investigate the case of no damping ($b = 0$) a bit more carefully:

$$mx'' + kx = F_{ext}(t).$$

Let's define the parameter

$$\omega_n = \sqrt{k/m}$$

and use this to rewrite the equation in terms of m and ω_n:

$$m(x'' + \omega_n^2 x) = F_{ext}(t).$$

We saw this form of the equation in the previous session. Recall that the subscript "n" stands for "natural". To remind ourselves why, consider the solution in the case of no driving force, i.e. $F_{ext}(t) = 0$. The characteristic equation $p(r) = m(r^2 + \omega_n^2)$, has roots $\pm i\omega_n$. Thus, the general solution is

$$x_h = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t).$$

So even without a driver, if we give the system a nudge it will oscillate at it's natural frequency ω_n.

Now let's add some sinusoidal input: $F_{ext}(t) = B\cos(\omega t)$. Using complex replacement, we must find a particular solution to

$$m(z'' + \omega_n^2 z) = Be^{i\omega t}.$$

Applying the exponential response formula with $a = i\omega$, we get

$$z_p = \frac{B}{p(i\omega)} e^{i\omega t} = \frac{B}{m(\omega_n^2 - \omega^2)} e^{i\omega t}.$$

Taking the real part,

$$x_p = \Re(z_p) = \frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t)$$

is a particular solution. All in all, our general solution is

$$x = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{B}{m(\omega_n^2 - \omega^2)} \cos(\omega t).$$

Again, the gain is given by $1/|p(a)| = 1/|p(i\omega)|$.

Example. Let's take $m = 1$, $B = 1$ and $\omega_n = 2$, and investigate the resulting particular solution as we vary ω, the frequency of the driving. The following figure shows the situation for $\omega = 3, 2.5$ and 2.1.

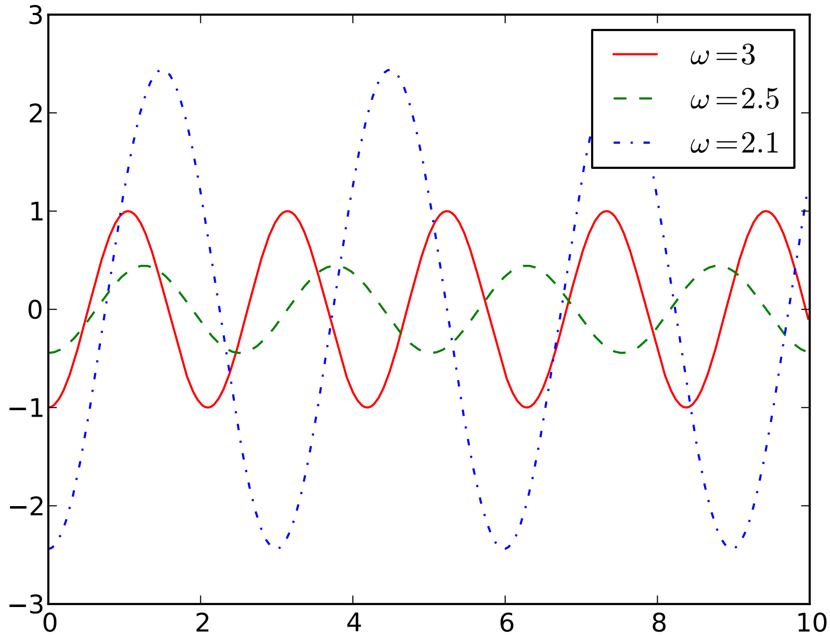

Fig. 1. Solutions for different values of ω.

What do you think happens as ω approaches the natural frequency?

It's no surprise that the solution breaks down when $\omega = \omega_n$. This situation is called **pure resonance** and we will investigate it in detail in an upcoming session. Notice that it corresponds to the case $p(a) = 0$ in the exponential response formula, since $p(i\omega_n) = 0$. For now, we'll note that it can be checked that the following is a solution:

$$x_p(t) = \frac{B}{2m\omega} t \sin(\omega t).$$

Notice that the extra factor of t before the sine term. So the amplitude grows with time, as shown in the following figure.

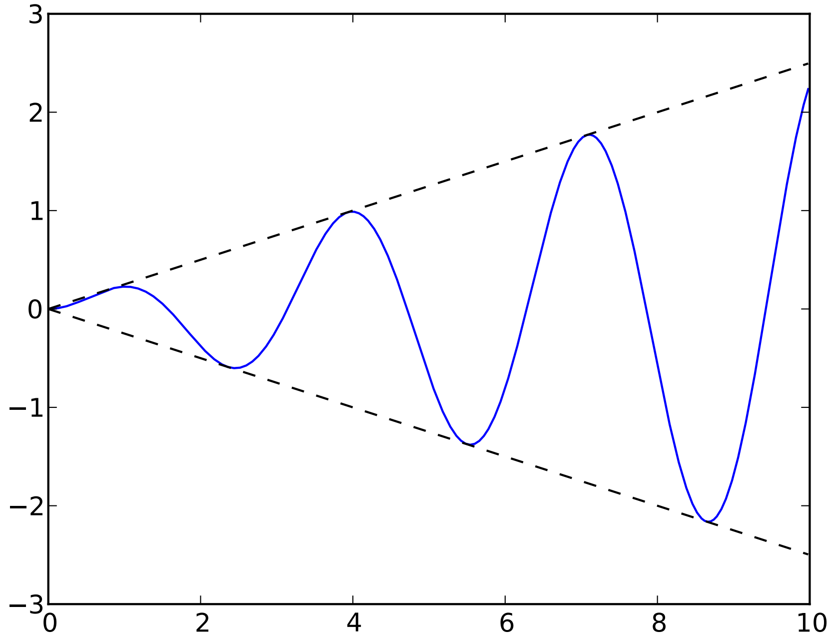

Fig. 2. Solution at pure resonance.

More generally, the following is the counterpart to the exponential response formula in the pure resonance case, when $p(a) = 0$.

Resonant Response Formula (RRF). Consider the second order equation

$$mx'' + kx' + bx = Be^{at},$$

with characteristic polynomial p. Then if $p(a) = 0$ and $p'(a) \neq 0$, then

$$x(t) = \frac{B}{p'(a)} te^{at}$$

is a particular solution.

Once we develop a small amount of algebraic machinery we will be able to give a simple proof of this formula.