Pset 10 Part I

For the following systems, the origin is clearly a critical point. Give its geometric type and stability, and sketch some nearby trajectories of the system.

Problem 1:

$$x' = x - y + xy$$
$$y' = 3x - 2y - xy$$

Solution: The Jacobian is $J(0,0) = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$.

(The linearization at $(0,0)$ is $x' = x - y$ and $y' = 3x - 2y$.)

The characteristic equation: $\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = \dfrac{-1 \pm \sqrt{-3}}{2}$.

Thus we have a spiral sink, which is asymptotically stable, at $(0,0)$.

Problem 2:

$$x' = x + 2x^2 - y^2$$
$$y' = x - 2y + x^3$$

Solution: The Jacobian is $J(0,0) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$.

(The linearization at $(0,0)$ is $x' = x$ and $y' = x - 2y$.)

The eigenvalues are $1, -2$ (these are easy to see because the matrix is triangular).

Thuse we have a saddle, which is unstable, at $(0,0)$

For the following systems carry out the linearization for sketching trajectories. Find the critical points, analyze each, draw in nearby trajectories , then add some other trajectories compatible with the ones you have drawn; when necessary, sketch in a well-chosen vector from the vector field to help.

Problem 3:

$$x' = 1 - y$$
$$y' = x^2 - y^2$$

Solution: The critical points are where $1 - y = 0$ and $x^2 - y^2 = 0$.
The first equation implies $y = 1$. The second then gives $x = \pm 1$.
Thus, we have critical points at $(1,1)$, $(-1,1)$.

In general the Jacobian matrix (of partial derivatives) is $J(x,y) = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$.

At (1,1) the linearization is $\begin{bmatrix} u' \\ v' \end{bmatrix} = J(1,1)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$.

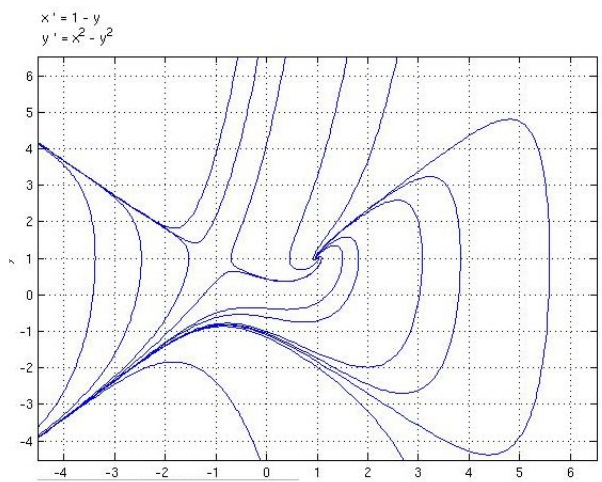
Characteristic equation: $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm \frac{\sqrt{-4}}{2} = -1 \pm i \Rightarrow$ asymptotically stable spiral sink near $(1,1)$

At $(-1,1)$ linearizing (again using Jacobian) : $\begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$

Characteristic equation: $\lambda^2 + 2\lambda - 2 = 0 \Rightarrow \lambda = -1 \pm \sqrt{3} \approx -2.73,\ 0.73$
$\Rightarrow$ unstable saddle near $(1,-1)$

For eigenvalue λ the eigenvector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ satisfies $-\lambda a_1 - a_2 = 0 \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\lambda \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.73 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2.73 \end{bmatrix}$

Using this information we get a sketch similar to the following computer-generated plot:



Problem 4:

$$x' = x - x^2 - xy$$
$$y' = 3y - xy - 2y^2$$

Solution: The critical points are where $x(1 - x - y) = 0$ and $y(3 - 2y - x) = 0$.
The first equation implies, either $x = 0$, or $1 - x - y = 0$.
If $x = 0$, the second equation gives: $y = 0$ or $y = 3/2$
If $1 - x - y = 0$, the second equation gives either $y = 0$, in which case $x = 1$, or $3 - 2y - x = 0$ in which case we solve the 2 equations

$$1 - x - y = 0$$
$$3 - 2y - x = 0$$

getting $y = 2, x = -1$

In summary, the four critical points are $(0, 0)$, $(0, 3/2)$, $(1, 0)$, $(-1, 2)$.

Now we determine their types:

The Jacobian matrix is $J = \begin{bmatrix} 1 - 2x - y & -x \\ -y & -x + 3 - 4y \end{bmatrix}$.

At (0,0): $J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

This has eigenvalues $1, 3$ with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\Rightarrow$ this is an unstable nodal source.

At $(0, \frac{3}{2})$: $J = \begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$.

This has eigenvalues $-1/2, -3$ with eigenvectors $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\Rightarrow$ stable nodal sink.

At (1,0): $J = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$.

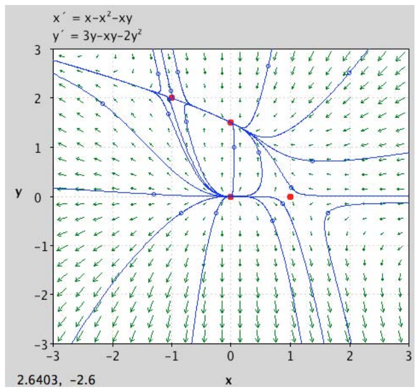
This has eigenvalues -1, 2 with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\Rightarrow$ unstable saddle.

At (-1,2): $J = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$.

Characteristic equation: $\lambda^2 + 3\lambda - 2 = 0 \Rightarrow \lambda = \frac{-3 \pm \sqrt{17}}{2} \approx 1/2, -7/2$

The corresponding eigenvectors are approximately $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 5/2 \end{bmatrix}$ $\Rightarrow$ unstable saddle.

Combining, we get a sketch of the trajectories similar to the computer generated figure at below.



Problem 5: *Structural stability:*
The following system has a critical point at the origin:

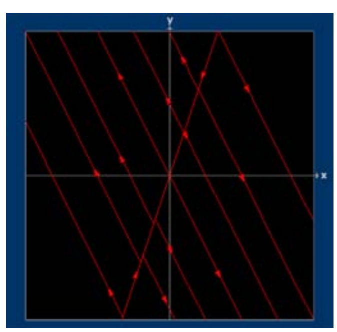
$$x' = 3x - y + x^2 + y^2, \qquad y' = -6x + 2y + 3xy.$$

For that critical point, find the geometric type and stability of the corresponding linearized system, and then tell what the possibilities would be for the corresponding critical point of the given non-linear system.

Solution: The linearization at (0,0) is $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 3 & -1 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ at $(0,0)$.

Characteristic equation: $\lambda^2 - 5\lambda = 0, \Rightarrow \lambda = 0, 5.$

Thus, in the linear system, (0,0) is not isolated - it is *one* of a line of critical points, all unstable. The phase portrait is



For the non-linear system, the picture could stay like this or it could be an unstable node or saddle.

Problem 6: *Structural stability:*
The following system has one critical point whose linearization is not structurally stable:

$$x' = y, \qquad y' = x(1-x).$$

Begin by finding the critical points and determining the type of the corresponding linearized system at each of the critical points. Then in each case, sketch several pictures showing the different ways the trajectories of the non-linear system might *look.*

Solution: The Jacobian is $J = \begin{bmatrix} 0 & 1 \\ 1-2x & 0 \end{bmatrix}$.

Critical points : $(0,0), (1,0)$

At $(0,0)$, $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Characteristic equation: $\lambda^2 - 1 = 0 \;\Rightarrow\; \lambda = 1, -1$, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

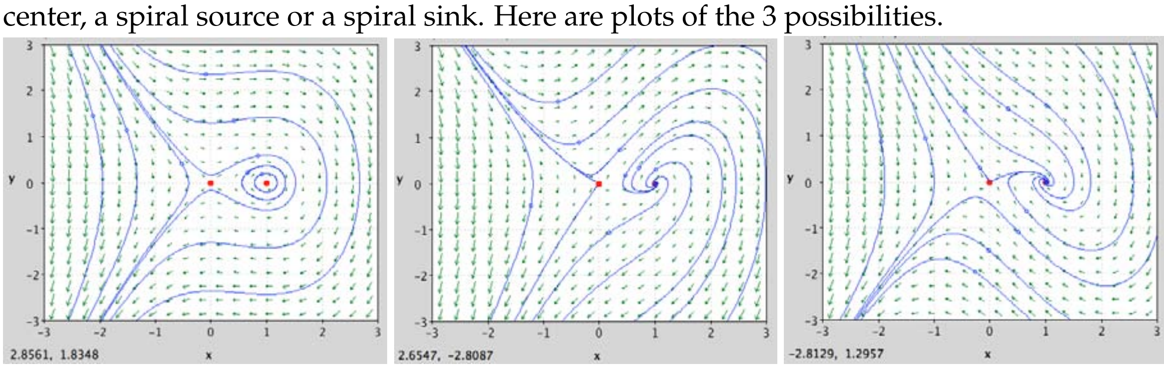
This is unstable saddle.

At $(1,0)$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Characteristic equation: $\lambda^1 + 1 = 0 \;\Rightarrow\; \lambda = \pm i$.

This is a *center*, with *clockwise motion*.

The linearized center is not structurally stable. At (1,0) the nonlinear system has either a center, a spiral source or a spiral sink. Here are plots of the 3 possibilities.



For the next three problems, find and classify the critical points of the given non-linear system. Then use this information to give a rough sketch of the solution curves.

Problem 7:

$$\begin{aligned} x' &= 60x - 4x^2 - 3xy \\ y' &= 42y - 2y^2 - 3xy \end{aligned}$$

Solution: Critical points: $(0,0)$, $(0,21)$, $(15,0)$, $(6,12)$.

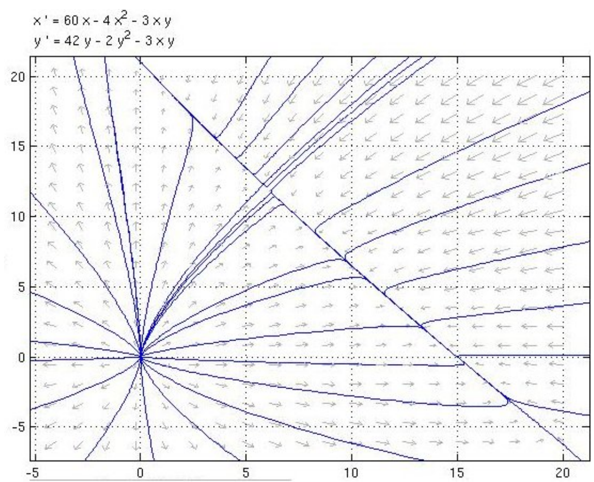
$J(x,y) = \begin{pmatrix} 60 - 8x - 3y & -3x \\ -3y & 42 - 4y - 3xy \end{pmatrix}$.

i) $J(0,0) = \begin{pmatrix} 60 & 0 \\ 0 & 42 \end{pmatrix} \Rightarrow \lambda = 60,\ 42 \Rightarrow$ nodal source.

ii) $J(0,21) = \begin{pmatrix} -3 & 0 \\ -63 & -42 \end{pmatrix} \Rightarrow \lambda = -3,\ -42 \Rightarrow$ nodal sink.

iii) $J(15,0) = \begin{pmatrix} -60 & -45 \\ 0 & -3 \end{pmatrix} \Rightarrow \lambda = -60,\ -3 \Rightarrow$ nodal sink.

iv) $J(6,12) = \begin{pmatrix} -24 & -18 \\ -36 & -24 \end{pmatrix} \Rightarrow \lambda = -24 \pm 18\sqrt{2} \Rightarrow$ saddle



Problem 8:

$$\begin{aligned} x' &= 5x - x^2 - xy \\ y' &= -2y + xy \end{aligned}$$

Solution: Critical points: $(0,0)$, $(5,0)$, $(2,3)$.

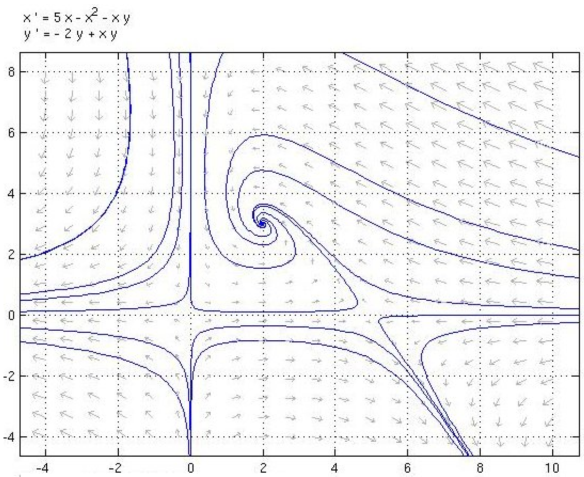
$$J(x,y) = \begin{pmatrix} 5-2x-y & -x \\ y & -2+x \end{pmatrix}.$$

i) $J(0,0) = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \lambda = 5, -2 \Rightarrow$ saddle.

ii) $J(5,0) = \begin{pmatrix} -5 & -5 \\ 0 & 3 \end{pmatrix} \Rightarrow \lambda = -5, 3 \Rightarrow$ saddle.

iii) $J(2,3) = \begin{pmatrix} -2 & -2 \\ 3 & 0 \end{pmatrix}.$

Characteristic equation: $\lambda^2 + 2\lambda + 6 = 0 \Rightarrow \lambda = -1 \pm \sqrt{5}\, i \Rightarrow$ spiral sink.



Problem 9:
$$\begin{aligned} x' &= x^2 - 2x - xy \\ y' &= y^2 - 4y + xy \end{aligned}$$

Solution: Critical points: $(0,0)$, $(0,4)$, $(2,0)$, $(3,1)$.

$$J(x,y) = \begin{pmatrix} 2x-2-y & -x \\ y & 2y-4+x \end{pmatrix}.$$

i) $J(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} \Rightarrow \lambda = -2, -4 \Rightarrow$ nodal sink.

ii) $J(0,4) = \begin{pmatrix} -6 & 0 \\ 4 & 4 \end{pmatrix} \Rightarrow \lambda = -6, 4 \Rightarrow$ saddle.

iii) $J(2,0) = \begin{pmatrix} 2 & -2 \\ 0 & -2 \end{pmatrix} \Rightarrow \lambda = 2, -2 \Rightarrow$ saddle.

iv) $J(3,1) = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix}$.

Characteristic equation: $\lambda^2 - 4\lambda + 6 = 0 \Rightarrow \lambda = 2 \pm \sqrt{2}\,i \Rightarrow$ spiral source.