18.03SC Practice Problems 8

Exponential and Sinusoidal Input

Solution Suggestions

1. *Find a solution of $\dot{x} + 2x = e^t$ of the form we^t. Do the same for $\dot{z} + 2z = e^{2it}$. (First determine the appropriate corresponding form of the solution.) In both cases, find the general solution.*

First we want to find a w so that we^t satisfies the first equation. This means w must satisfy $\dot{w}e^t + we^t + 2we^t = e^t$, or, equivalently, $\dot{w} + 3w = 1$. This equation is satisfied by the constant $w = \frac{1}{3}$, so a particular solution is

$$x_p(t) = \frac{1}{3}e^t.$$

For the second part, we want to find a solution of the form ve^{2it} to match the form of the input function in the second equation. The condition for v is $\dot{v} + (2i+2)v = 1$, which is satisfied by the constant $v = \frac{1}{2+2i}$. Hence, a particular solution of the desired form is

$$z_p(t) = \frac{1}{2+2i}e^{2it}.$$

There are several ways to find the general solutions. We could solve each equation directly by using the integrating factor method, getting $x = \frac{1}{3}e^t + ce^{-2t}$ for the first case and $z = \frac{1}{2+2i}e^{2it} + ce^{-2t}$ for the second case. However, a "better" way to solve this problem would be to use the fact that the general solution to a linear ODE is the sum of a particular solution, which depends on the input function, and the general homogeneous solution, which only depends on the system. This is a special case of the superposition principle, which might be slightly ahead of the material presented so far, but captures the right intuition for dealing with these types of problems.

These two equations have the same homogeneous part, $\dot{x} + 2x = 0$, or $\dot{x} = -2x$, with homogeneous solution $x = ce^{-2t}$. Remember that we just found a particular solution for each case. So general solutions are

$$x = \frac{1}{3}e^t + ce^{-2t} \quad \text{and} \quad z = \frac{1}{2+2i}e^{2it} + ce^{-2t}.$$

The point of this problem is to realize that the two equations model the same system driven by different input functions and that it may be easier to find general solutions given a particular, even when the input is complex-valued.

2. *Find a solution of $\dot{x} + 2x = \cos(2t)$ using complex replacement. Your work should also give you a solution for $\dot{x} + 2x = \sin(2t)$.*

Notice that $\cos(2t) = \text{Re}(e^{2it})$, so x can be the real part of any solution z to the complex valued equation $\dot{z} + 2z = e^{2it}$. In particular, from problem 1, one solution is given by $x = \text{Re}\left(e^{2it}/(2+2i)\right) = \frac{\cos(2t)+\sin(2t)}{4}$. A solution to $\dot{x} + 2x = \sin(2t)$ is the corresponding imaginary part, i.e., $x = \frac{-\cos(2t)+\sin(2t)}{4}$.