PROFESSOR: Welcome back. So in this session, we're going to look at sinusoidal inputs for ordinary differential equations of order one. So here in question one, you're asked to use complex techniques to solve x dot plus k*x equals cosine of omega*t. Here k and omega are constants. This is of ODE of first order. And the sinusoidal input is referring to the right-hand side of the ODE where basically the we are forcing a system with a function, and it's sinusoidal of angular frequency omega, which means basically a period of 2pi over omega.

So you're asked to use complex techniques to solve this in question a. In question b, we're asked to use what we had in question a to solve this modified function, where again, we have a sinusoidal input on the right-hand side with an additional amplitude F, which is now constant.

In the third part, we're asked to use superposition principle to solve this combined equation, where now we're introducing a value for example for F, which is 3. And you can see also that the right-hand side is also a sinusoidal input, because it's a linear combination of sinusoidal functions. All these equations are linear with constant coefficients, and hence superposition principle would hold. So why don't you stop the video, take a few minutes, and work through the problems. And we'll be back.

Welcome back. So what is it that we're asked to do here? We're asked to solve this using a trick that you learned in class to basically convert a real-valued ODE into a complex form. So the first thing to do is to realize that the cosine of omega*t is simply the real part of the imaginary complex exponential i*omega*t. So when we realize that, and we see that the ODE is real-valued linear, we can convert this real-value ODE into this complex-valued ODE.

And we're going to label this equation star. From this point, we can go back to the techniques that we learned in class, namely the integrating factor, to realize that we can rewrite the right-hand side in this form, introducing a new function u, so that we need to seek a function u that will then recover this function, this equation. And from previous recitations and problem, we saw that clearly the u that we will need to pick is just exponential of k*t.

So now from this, basically we're back to i*omega plus k, the whole thing t, on the right-hand side, and here z*u prime that we just need to integrate on both sides. So this is simply k*t*z on the left-hand side. And on the right-hand side, we're integrating-- again, it's still an exponential, even though it's complex. And we need to introduce, of course, constant of integration.

So the solution for the equation star is then, I'm going to write it up here, so that we keep that for the rest of the problem, minus t minus k*t when I divide by this. And here, the minus k*t of this exponential is going to be canceled out by the integrating factor. So we need to keep that here.

So here I gave the general solution with the solution that would come from the homogeneous equation, which we could refer to also as a transient here, because basically after a long time t, this exponential, if k is positive, would decay. And the part of the solution that comes from the sinusoidal input, or the forcing, which would be particular solution.

So to go back to the original question, we were asked to solve this real-valued equation. So from what we noticed above, we saw that the right-hand side is just a real part of the right-hand side of the complex value equation star. And similarly then, we can do the same for the actual solutions themselves. x is also just the real value of this complex number.

So here, for this general solution, I'm going to just use c bar for general complex constant here. And in this case, we will take the real value. So at this point, it doesn't really matter what t is. It's just we're going to keep it as a constant. And now it's just real valued. That's for the one part. And then we need to take the real part of this expression. Come back to the line here.

So to take the real part of this expression, you learned that basically you just need to multiply the denominator or the numerator by the complex conjugate. And that will only give us-- so we have k minus i*omega over the squares, and then again, Euler formula that we saw in a previous recitation.

So x, from this expression, is just the real part of all of this. So it's going to be this term. Let me factorize for a moment here, and then the result of multiplying the two complex parts as well, i to i, minus 1. And so that answers part a of the problem.

So if we look at part b-- I'm just going to do it here-- part b we're asked to do to solve a very, very similar equation. And I'm just going to leave the F out for now. It's basically the same equation, except that the input now is a sine instead of a cosine.

So we can use the same trick as we used for question a by realizing that now the sine is just an imaginary part of the exponential, the complex exponential. So we don't need to redo all the work. We only need to, if we were considering this equation, to just take the imaginary part of the solution that we just found here. And we can just read off the solution from this line.

Note that in this expression, I left out the homogeneous part, which I should add here from here. And this was just a complex part. So to come back to what I was saying, the sine is just the imaginary part of the complex exponential. So we're just going to write down the solution by reading off here. So again, we have the homogeneous part, which would be another real-valued constant, and then the real part of this expression, 1 over k squared plus omega squared. And its imaginary part then would give us a k sine omega*t and a minus omega cos omega*t. So this would be the solution for part b, if we had this equation.

But we actually have an equation with an additional amplitude F. And what this leads to do is just an additional constant that would appear then for this solution. And we could see that if we rewound the way we solved in part a, and saw that introducing constant F to the equation is equivalent to just multiplying the full equation with F. And then in the integrating factor part, we would end up with an F in front of our solution when we are doing the integration. And so basically, that amplitude could be absorbed in the constant of integration for the homogeneous part. And the F just remains in the particular solution here.

So for the last part of the problem, we were now asked to use the superposition principle to combine solutions of two previous ODEs that we could split from this equation, cosine omega*t plus 3 sine omega*t. So you saw from previous recitations that superposition principle applies for linear equations. This is clearly a linear equation. And here on the right-hand side, we have two sinusoidal functions. Because it's linear we can look at this by splitting it into two equations. And I'm just introducing x_1, x_2 as notation to distinguish between the two.

And here we can recognize that we already did this work that was in part a, and we already did this work in part b, where F now has a value of 3. And so what this is telling us, because it's a linear equation, we can use superposition principle, the solution, if you were to do the addition of these two equations, would just be the sum of the previous solutions that we found. So this would be our x_3, if I had labeled this with a 3.

And so basically, the solution is simply the sum of the two previous forms. And so we would end up still with our c minus k*t, which is this basically the homogeneous part that we obtained in the two previous parts, which is common. And then we just need to add each particular solution introduced in part a and part b. So if I have room I will write it all out.

So we would have k cos omega*t plus omega sine omega*t from the first part of the cosine.

And then we're introducing a 3k sine omega*t minus 3 omega cosine omega*t. And that would be then the solution of the combined two equations.

And so you don't need to redo the full work, by using the superposition principle. So here the key was just to recognize that the cosines and the sines are basically real parts and imaginary parts of the complex number, of the exponential of i*omega*t. Again, it's just Euler formula. And using that as a shortcut to be able to kill two birds with one stone and solve the two equations by using only one approach, which is just the integrating factor.

And this ends the session for today.