Part II Problems and Solutions

Problem 1: *[Natural growth, separable equations] In recitation a population model was studied in which the natural growth rate of the population of oryx was a constant $k > 0$, so that for small time intervals Δt the population change $x(t + \Delta t) - x(t)$ is well approximated by $kx(t)\Delta t$. (You also studied the effect of hunting them, but in this problem we will leave that aside.) Measure time in years and the population in kilo-oryx (ko).*

A mysterious virus infects the oryxes of the Tana River area in Kenya, which causes the growth rate to decrease as time goes on according to the formula $k(t) = k_0/(a + t)^2$ for $t \geq 0$, where a and k_0 are certain positive constants.

(a) *What are the units of the constant a in "$a + t$," and of the constant k_0?*

(b) *Write down the differential equation modeling this situation.*

(c) *Write down the general solution to your differential equation. Don't restrict yourself to the values of t and of x that are relevant to the oryx problem; take care of all values of these variables. Points to be careful about: use absolute values in $\int \frac{dx}{x} = \ln|x| + c$ correctly, and don't forget about any "lost" solutions.*

(d) *Now suppose that at $t = 0$ there is a positive population x_0 of oryx. Does the progressive decline in growth rate cause the population stabilize for large time, or does it grow without bound? If it does stabilize, what is the limiting population as $t \to \infty$?*

Solution: **(a)** The growth rate $k(t)$ has units years^{-1} (so that $k(t)x(t)\Delta t$ has the same units as $x(t)$). The variable t has units years, so the a added to it must have the same units, and k_0 must have units years in order for the units of the fraction to work out.

(b) $x(t + \Delta t) \simeq x(t) + k(t)x(t)\Delta t$, so $\dot{x} = k_0 x/(a + t)^2$.

(c) Separate: $dx/x = k_0(a + t)^{-2}dt$. Integrate: $\ln|x| + c_1 = -k_0(a + t)^{-1} + c_2$. Amalgamate constants and exponentiate: $|x| = e^c e^{-k_0/(a+t)}$. Eliminate the absolute value: $x = Ce^{-k_0/(a+t)}$, where $C = \pm e^c$. Reintroduce the solution we lost by dividing by x in the first step: allow $C = 0$. So the general solution is $x = Ce^{-k_0/(a+t)}$. (Note that the exponent $-k_0/(a + t)$ is dimensionless, as an exponent must be.)

(d) When t gets very large, the exponent gets very near to zero, so there is a finite limiting population: $x_\infty = C$. Thus $x(t) = x_\infty e^{-k_0/(a+t)}$. Take $t = 0$ in the solution: $x_0 = x_\infty e^{-k_0/a}$, or $x_\infty = e^{k_0/a} x_0$.