Sinusoidal Input

The exponential response formula works perfectly even if the number a in the exponential is complex. Let's use this to solve problems with a sinusoidal driving.

Example. Find the general solution to

$$x'' + 8x' + 7x = 9\cos(2t).$$

We begin by using complex replacement and considering instead the equation

$$z'' + 8z' + 7z = 9e^{2it}. \tag{1}$$

Now we can apply the exponential response formula to obtain as a particular solution,

$$\begin{aligned} z_p(t) &= \frac{9}{p(2i)}e^{2it} \\ &= \frac{9}{(2i)^2 + 16i + 7}e^{2it} \\ &= \frac{9}{3 + 16i}e^{2it}. \end{aligned}$$

Be careful with signs when you do these calculations! Remember $i^2 = -1$.

To get a particular solution to (1), we must take the real part. We prefer the solution in amplitude-phase form, so we write

$$3 + 16i = \sqrt{265}e^{i\phi} \qquad \text{where} \qquad \phi = \tan^{-1}(16/3).$$

Thus (be careful not to forget the factor of 9 in the complex solution)

$$x_p(t) = \Re(z_p(t)) = \frac{9}{\sqrt{265}}\cos(2t - \phi).$$

To get the general solution we must add the general solution of the homogeneous problem, which we already saw:

$$x_h(t) = c_1e^{-7t} + c_2e^{-t}.$$

Thus we obtain the general solution

$$x = x_p + x_h = c_1e^{-7t} + c_2e^{-t} + \frac{9}{\sqrt{265}}\cos(2t - \phi).$$

Notice that in the example above, the *amplitude* of the particular solution is given by

$$A = \frac{9}{|p(2i)|} = \frac{9}{\sqrt{265}}.$$

If we consider the input to the system to be $9\cos(2t)$ then the input has amplitude 9 and the output amplitude is given by the input amplitude multiplied by $1/|p(2i)|$. This factor is called the **gain** of the system:

$$\text{output amplitude } = \text{gain} \times \text{input amplitude}.$$

Said differently, the gain is the ratio of the output amplitude to the input amplitude.

Let's apply the above sequence of steps to the general case of a sinusoidal driving:

$$mx'' + bx' + kx = B\cos(\omega t).$$

The complixified equation is

$$mz'' + bz + kz = Be^{i\omega t}.$$

From the exponential response formula with $a = i\omega$, a particular solution is

$$z_p = \frac{B}{p(i\omega)}e^{i\omega t}.$$

Converting to polar form and then taking the real part, we get

$$x_p = \frac{B}{|p(i\omega)|}\cos(\omega t - \phi),$$

where $\phi = \arg(p(i\omega))$. Notice that since $a = i\omega$, the gain is given by

$$1/|p(a)| = 1/|p(i\omega)|.$$

In later sections we will go through the notion of input, and gain more carefully.

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