18.034 Honors Differential Equations
Spring 2009

LECTURE 20. TRANSFORM AND DIFFERENTIAL EQUATIONS

Properties of Laplace transform. We derive several important properties of Laplace transforms.

Theorem 20.1. *Let $\mathcal{L}[f(t)](s) = F(s)$. For $f \in E$,*

(i) *(s-shift)* $\mathcal{L}[e^{-ct}f(t)] = F(s+c)$.
(ii) *(t-shift)* $\mathcal{L}[f(t-c)] = e^{-sc}F(s)$ *if* $c \geqslant 0$ *and* $f(t) = 0$ *for* $t < 0$.
(iii) *(s-derivative)* $\mathcal{L}[tf(t)] = -F'(s)$.
(iv) *(t-derivative)* $\mathcal{L}[f'(t)] = sF(s) - f(0)$, *if* f *is continuous.*
(v) *(scaling)* $\mathcal{L}[f(ct)] = \frac{1}{c}F\left(\frac{s}{c}\right)$, $\quad F(sc) = \frac{1}{c}\mathcal{L}\left[f\left(\frac{t}{c}\right)\right]$ *if* $c > 0$.

Proof. 1. It follows from

$$\int_0^\infty e^{-st}e^{-ct}f(t)\,dt = \int_0^\infty e^{-(s+c)t}f(t)\,dt.$$

2. Let $u = t - c$. Then,

$$\int_0^\infty e^{-st}f(t-c)\,dt = \int_{-c}^\infty e^{-s(u+c)}f(u)\,du = \int_0^\infty e^{-sc}e^{-su}f(u)\,du.$$

The limits $(-c, \infty)$ can be changed to $(0, \infty)$ by hypothesis.

3. It follows from

$$\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st}f(t)\right)\,dt = \int_0^\infty -te^{-st}f(t)\,dt.$$

4. It is proved in Lecture 19.
5. By change of variables. □

Exercise. Show that

1. $\mathcal{L}\left[\dfrac{f(t)}{t}\right] = \displaystyle\int_s^\infty F(s)\,ds$ if $f(t)/t \in E$.
2. $\mathcal{L}\left[\displaystyle\int_0^t f(t)\,dt\right] = \dfrac{F(s)}{s}$ if $f \in E$.

Example 20.2. Compute the Laplace transform of te^t.

SOLUTION. Applying the s-derivative property to $\mathcal{L}[e^t] = \dfrac{1}{s-1}$ gives

$$\mathcal{L}[te^t] = -\left(\frac{1}{s-1}\right)' = \frac{1}{(s-1)^2}.$$

Exercise. Show that $\mathcal{L}[t^n e^{at}] = \dfrac{n!}{(s-a)^{n+1}}$, $\quad n = 0, 1, 2, \ldots$ and $a \in \mathbb{R}$.

Example 20.3. Compute the Laplace transform of $e^{3t}\sin t$.

SOLUTION. Applying the s-shift property to $\mathcal{L}[\sin t] = \dfrac{1}{s^2+1}$ gives

$$\mathcal{L}[e^{3t}\sin t] = \frac{1}{(s-3)^2+1}.$$

Exercise. Show that $\mathcal{L}[e^{-ct}\cos bt] = \dfrac{s+c}{(s+c)^2+b^2}$ and $\mathcal{L}[e^{-ct}\sin bt] = \dfrac{b}{(s+c)^2+b^2}$.

Example 20.4. What function has Laplace transform $\dfrac{4s}{(s^2+4)^2}$?

SOLUTION. Observe that

$$\left(\frac{2}{s^2+4}\right)' = \frac{-4s}{(s^2+4)^2} \quad \text{and} \quad \mathcal{L}[\sin 2t] = \frac{2}{s^2+4}.$$

Hence, by the s-derivative property,

$$\mathcal{L}^{-1}\left[\frac{4s}{(s^2+4)^2}\right] = t\sin 2t.$$

Example 20.5. What function has Laplace transform $\dfrac{1}{s^2+4s+9}$?

SOLUTION. It involves completing the square. We write

$$\frac{1}{s^2+4s+9} = \frac{1}{(s+2)^2+5}.$$

Since $\mathcal{L}\left[\sin\sqrt{5}t\right] = \dfrac{\sqrt{5}}{s^2+5}$, by the s-shift property

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+9}\right] = \frac{1}{\sqrt{5}}e^{2t}\sin\sqrt{5}t.$$

Exercise. Show that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+4s+9}\right] = \frac{1}{\sqrt{5}}e^{2t}\cos\sqrt{5}t + \frac{2}{\sqrt{5}}e^{2t}\sin\sqrt{5}t.$$

Generalized solutions. We study

$$P(D)y := y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = f(t), \tag{20.1}$$

where a_j are constants and $f \in E$. That is, f is allowed to have discontinuities. a very effective method of dealing with such problems. First, we must come to grips with the following.

Theorem 20.6. *Let $n \geqslant 1$ and $t \in I$ be an open interval. If f has a simple discontinuity at some point in I, then (20.1) has no classical solution on I.*

At a *simple discontinuity* the left limit and the right limit exist but they do not agree. A classical solution on I is a function $y = \phi(t)$ which satisfies the differential equation at every point of I. The condition $n \geqslant 1$ ensures that (20.1) actually involves differentiation. The proof uses a theorem of Darboux and it is omitted here.

Here we extend the notion of "solution" to allow discontinuous inputs and we develop the theory of the Laplace transform within the context of the extension.

Definition 20.7. *A function $y = \phi(t)$ is called a* generalized solution *of (20.1) on the interval I if*

(i) *$\phi, \phi', \ldots, \phi^{(n-1)}$ are continuous on I, and*
(ii) *$P(D)\phi(t) = f(t)$ on I wherever f is continuous.*

The condition (i) means all bad behaviors of f are absolved by $\phi^{(n)}$. At discontinuities of f, the equation may fail. $y^{(n)}$ doesn't need to exist.

Exercise. Establish the existence and uniqueness theorem of the initial value problem of (20.1) and $y(0) = y_0, y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1}$, where $f \in E$ in the class of generalized solutions.

Theorem 20.8. *If y is a generalized solution of* (20.1) *on $t \in [0, \infty)$ and $f \in E$, then $y, y', \dots, y^{(n)} \in E$.*

Sketch of proof. First, show that the class E is closed under addition, multiplication, and integration. That is, if $f, g \in E$, then $f + g, fg, \int f, \int g \in E$.

Next, if $P(D) = D - a$, then the solution of $P(D) = f$ is given by

$$y(t) = e^{at} \int_0^t e^{-as} f(s)\, ds + ce^{at}.$$

Therefore, if $f \in E$ then $y \in E$. To show that the derivative belongs to E, we write the equation as $y' = f - p_0 y$. Since $f, y \in E$, so is y'.

Finally, we use induction on n. $\square$

The transformed equation. Let us consider

$$P(D)y := y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = f \in E, \tag{20.2}$$

where a_j are constants. The characteristic polynomial of $P(D)$ is

$$P(s) := s^{(n)} + a_1 s^{(n-1)} + \cdots + a_n.$$

When f has a discontinuity, then the word "solution" will be used to mean "generalized solution". Using the formula

$$\mathcal{L}[f^{(n)}(t)](s) = s^n \mathcal{L}[f(t)](s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$$

recursively, the Laplace transform of (20.2) gives,

$$s^n \mathcal{L}y - (s^{n-1} y_0 + \cdots + y_{n-1}) + a_1 s^{n-1} \mathcal{L}y - a_1 (s^{n-2} y_0 + \cdots + y_{n-2}) + \cdots + a_n \mathcal{L}y = \mathcal{L}f,$$

which can be written as

$$P(s)Y(s) = F(s) + P_0(s),$$

where $Y(s) = \mathcal{L}y$, $F(s) = \mathcal{L}f$, and P_0 is a polynomial of degree $\leqslant n - 1$ whose coefficients depend on the initial conditions. A formula of P_0 is readily obtained by (19.4). Therefore, we arrive that

$$Y(s) = \frac{F(s)}{P(s)} + \frac{P_0(s)}{P(s)}.$$

Then, by finding the inverse transform of $Y(s)$ we obtain the solution of (20.2).

Suppose that the inhomogeneous term $f(t)$ is a finite sum of functions of the form

$$t^m e^{at} (A \cos bt + B \sin bt),$$

where $m \geqslant 0$ and a, b, A, B are constants. Inputs of this type are encountered frequently and in diverse contexts. The Laplace transform of $f(t)$ is a sum of rational functions, that is, quotients of two polynomials. For example, $\mathcal{L}[t^m e^{at}] = m!/(s - a)^{m+1}$. The discussion above then tells us that the Laplace transform of the output $y(t)$ is again a sum of rational functions.

When $F(s)$ is a rational function, the basic method of recovering $f(t)$, where $\mathcal{L}f(t) = F(s)$, is by expanding $F(s)$ into partial fractions.

Example 20.9. Solve the initial value problem

$$y'' - 2y' + 2y = 2e^t, \qquad y(0) = 0, \quad y'(0) = 1$$

By means of the Laplace transform.

SOLUTION. Let $Y(s) = \mathcal{L}y$ and $F(s) = \mathcal{L}f$. Taking the transform, we obtain

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = 2\frac{1}{s-1}, \qquad Y(s) = \frac{s+1}{(s-1)(s^2-2s+2)}.$$

We expand the rational function in partial fractions and then complete the square to obtain

$$\begin{aligned}\frac{s+1}{(s-1)(s^2-2s+2)} &= \frac{2}{s-1} + \frac{-2s+3}{s^2-2s+2} \\ &= \frac{2}{s-1} - 2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}\end{aligned}$$

By finding the inverse transform of each term,

$$y(t) = 2e^t - 2e^t \cos t + e^t \sin t.$$

Exercise. (The initial and the final value theorems) 1. If $f \in E$, show that $\lim\limits_{s\to\infty} F(s) = 0$.
2. If $f' \in E$ and f is continuous, show that $\lim\limits_{s\to\infty} sF(s) = f(0)$.
3. If $f \in E$ and $\lim\limits_{t\to\infty} f(t) = k$, show that $\lim\limits_{s\to 0+} sF(s) = k$.