18.034, Honors Differential Equations
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Notes from Lecture 36, 5/5/04

I. The winding number

Let $R \subset \mathbb{R}^2$ be an open region, let $\begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$

be an autonomous differential system on R, and let $C \subset R$ be an oriented, simpler closed curve in R. In other words, C is the image of the circle under a 1-to-1 map whose derivative vector is always nonzero, say $h : [0,1] \to R, h(1) = h(0)$.
If C contains no equilibrium point of the system, the following function is well-defined and continuous:

$$f : C \to \delta^1 \subset \mathbb{R}^2, \quad f(q) = \frac{1}{\sqrt{F(q)^2 + G(q)^2}} \begin{bmatrix} F(q) \\ G(q) \end{bmatrix}.$$

The composition $foh : [0,1] \to \delta^1$ is (essentially) a cts. map from the circle to the circle. To such a map there is an associated integer n, the degree of the map. This integer counts the number of times $foh(t)$ rotates counterclockwise around the circle as t rotates once counterclockwise around the circle. If h, F and G are all continuously differentiable function, $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$, and then the degree is simply

$$\frac{1}{2\pi} \int_0^1 \left[g_1(t) g_2'(t) - g_1'(t) g_2(t) \right] dt$$

This integer turns out to be independent of $h(t)$ (although it does depend on the orientation of C). it is called the winding number of (F,G) about C.

Let $p \in R$ be an equilibrium point. It is isolated if there exists $\varepsilon > 0$ such that p is the only equilibrium point in the ε- ball about p. For any $0 < q < \varepsilon$, consider the circle C_q of radius q centered at p. The winding number of $(F,6)$ about C_q is independent of q and is called the index of $(F,6)$ at p (or sometimes the Poincare index).

Examples: (1) Let λ, $\mu > 0$ and let $F = \lambda x, G = \mu y$. Then $p = (0,0)$ is an isolated equilibrium point. Consider $h_q(t) = \begin{bmatrix} q.\cos(t) \\ q.\sin(t) \end{bmatrix}$, $0 < t$

Then $g(t) = \frac{1}{\sqrt{\lambda^2 \cos^2(t) + \mu^2 \sin^2(t)}} \begin{bmatrix} \lambda \cos(t) \\ \mu \sin(t) \end{bmatrix}$.

And $g_1(t)g_2'(t) - g_1'(t)g_2(t) = \dfrac{\lambda\mu}{\lambda^2\cos^2(t)+\mu^2\sin^2(t)}$. This is closely related to the Poisson kernel. It is nontrivial, but the integral $\int_0^{2\pi} \dfrac{\lambda\mu}{\lambda^2\cos^2(t)+\mu^2\sin^2(t)}dt$ can be computed by elementary methods, and it equals 2π (consider the case that $\lambda = \mu$). So the index is +1.

(2) λ, $\mu < 0$. This is the same as above when $\lambda \to -\lambda$, $\mu \to -\mu$. Notice the integral does not change. So the index is +1.

(3) λ <0, μ >0. now the integral is $-\int_0^{2\pi} \dfrac{(-\lambda)\mu}{(-\lambda)^2\cos^2(t)+\mu^2\sin^2(t)}dt$.

This is -1 times the integral from (1). So the index is -1.

Theorem: Let C be a simple closed curve that contains no equ. pts. in R oriented so the interior is always on the left. If the interior is contained in R, and if the interior contains only finitely many equilibrium points, $p_1, \ldots, p_n$, then the winding number about C is index $(p_1) + \ldots + \text{index}(p_n)$. (and 0 if there are no eq. pts).

Proof: This is proved, for instance in Theorem 3, § 11.9 on p.442 of Wilfred Kaplan, Ordinary Differential Equations, Addison-Wesley, 1958.

Corollary: If R is simply-connected, then every cycle C contained in R contains an equilibrium point in its interior.

Rmk: A region R in $\mathbb{R}^2$ is simply-connected if for every simple closed curve C in R, the interior of C is contained in R. A cycle is a periodic orbit (that is necessarily a simple closed curve).

Pf: By construction, $(F,6)$ is parallel to the tangent vector of C. Therefore the winding number is +1. So, by the theorem, there is an equilibrium point in the interior of C.

II. Lyapunov functions

Let $R \subset \mathbb{R}^n$ be an open region. Let $\vec{x}' = \vec{F}(\vec{x})$ be an autonomous system on R. Let $p \in R$ be a point.

Definition: A function $V: R \to \mathbb{R}$ is positive definite (resp. negative definite) if
- (1) $V(q) \geq 0$ (resp. $V(q) \leq 0$) for all $q \in R$
- (2) $V(q) = 0$ iff $q = p$.

Let p be an equilibrium point.

Definition: A strong Lyapunov function is a continuously differentiable function $V : R \to \mathbb{R}$ such that

(1) V is positive definite

(2) the function $V' := \sum_{i=1}^{n} \frac{dV_{(x)}}{dx_i} F_i(x)$ is negative definite.

Remark : It is often the case that there is no strong Lyapunov function on R, yet there is an open subregion $R' \subset R$ containing p and a strong Lyapunov function on R'. In this case, simply replace R by R' in what follows.

Hypothesis: Suppose a strong Lyapunov function exists. There is a minor issue that your book does not deal with: long-time existence of solution curves. Let $K \subset \mathbb{R}^n$ be a bounded closed region whose interior contains p and such that $K \subset R$. Define r_0 = minimum of V on the bounded closed set ∂K (a continuous function on a bounded closed subset of $\mathbb{R}^n$ always attains a minimum). Because $p \in$ interior of K, $r_0 > 0$. Define R' to be

$$R' = KnV^{-1}\left([0, r_0] \right) = \left\{ q \in K / V_{(q)} < r_0 \right\}.$$

Observe this is an open region in R that contains p and is contained in the interior of K.

Theorem:(1) For every $x_0 \in R'$, the solution curve $x(t)$ is defined for all $t > 0$.

(2) Moreover, $\lim_{t\to\infty} x(t) = p$. Therefore p is an attractor and R' is in the basin of attraction of p.

Proof: For any $x_0 \in R$, if $x(t)$ is defined on the interval $[0, t_1)$, consider $V(x(t))$ defined on $[0, t_1)$. By the Chain Rule, $V(x(t))$ is differentiable and

$$\frac{d}{dt} V(x(t)) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t)) \cdot \frac{dx_i}{dt}(t).$$

By hypothesis, $x_i'(t) = F_i(x(t))$. Thus $\frac{d}{dt} V(x(t)) = V'(x(t))$.

By hypothesis, this is nonpositive. Therefore $V(x(t))$ is a non-increasing function. In particular, if $x_0 \in R'$, then $x(t)$ is in R' for all $t \in [0, t_1)$.

(1) Let $x_0 \in R'$. By way of contradiction, suppose that $x(t)$ is defined only on $[0, t_1)$ where t_1 is finite. By the theorem on maximally extended solutions, $\lim_{t\to t_1} x(t)$ exists and is in ∂K. Therefore $V(\lim_{t\to t_1} x(t)) \geq r_0$. Since V is continuous
$V(\lim_{t\to t_1} x(t)) = \lim_{t\to t_1} V(x(t))$. For all $t \geq 0$, $V(x(t)) \leq V(x_0) < r_0$.

So $\lim_{t\to t_1} V(x(t)) \leq V(x_0) < r_0$. This contradiction proves $x(t)$ is defined for all t>0.

(2) Let ε>0 and let $B_\varepsilon(p)$ denote the open ball of radius ε centered at p. The set difference $K \setminus (KnB_\varepsilon(p))$ is closed and bounded. Therefore V attains a mimimum value r_1 on this set. Since p is not in this set $r_1 > 0$. Also, $KnV^{-1}([r_1,\infty))$ is a closed set contained in K. So it is closed and bounded (K is bounded). Therefore V' attains a maximum value $-m_1$ on this set. Since p is not in this set $-m_1 < 0$, i.e. $m_1 > 0$.

Define $t_1 = \dfrac{r_0 - r_1}{m_1}$.

The claim is that for all $x_0 \in R'$, $V(x(t)) < r_1$ for all. In particular, since $x(t) \in R'$ & $V(x(t)) < r_1, x(t)$ is in R' n $B_\varepsilon(p)$. By way of contradiction, suppose $V(x(t)) \geq r_1$. By the mean value theorem, there exists t' with $0 < t' < t$ such that $V(x_0) - V(x(t)) = -V'(x(t')) \cdot t$. Since $V(x(t)) \geq r_1$, also $V(x(t')) \geq r_1$. Therefore $x(t')$& $KnV^{-1}[[r_1,\infty)]$.

Thus - $V'(x(t)) \geq m_1$. So $V(x_0) - V(x(t)) \geq m_1 t > m_1 t_1 = r_0 - r_1$. But $V(x_0) < r_0$ and $V(x(t)) \geq r_1$. This is a contradiction, proving $V(x(t)) < r_1$ for all $t > t_1$.

The definition of a weak Lyapunov function as well as the statements of Lyapunov's second and third theorems are in the textbook.

III. A criterion for asymptotic stability.

Let V be a real vector space of dimension n, eg. $\mathbb{R}^n$. Let $R \subset V$ be an open region, and let $\vec{x}' = \vec{F}(\vec{x})$ be an autonomous system on R. Let $p \in R$ be an equilibrium point.

Theorem: If F is differentiable at p, and if every eigenvalue of $\left[\dfrac{\partial F_i}{\partial R_j}\right]_p$ has negative real part, then there is an open region $R' \subset R$ contains p and a story Lyapunov function on R'.

Proof:There is a beautiful proof in the first edition of the textbook, which is stapled at the end. Here we give a closely related, but different argument.

The Jacobian of F at p is a linear transformation $T: \mathbb{V} \to \mathbb{V}$ with the property that, for my norm $\|-\|$ on $\mathbb{V}$, for every $\varepsilon > 0$, $\exists\, \partial^2 > 0$ such that if $\|V\| < \partial^2$, then $\|F_{(p+v)} - F_{(p)} - T_v\| \leq \varepsilon \cdot \|V\|$. Notice this is independent of the system of coordinates on $\mathbb{V}$. Without loss of generality, translate so $p = 0$.

As we have alluded to earlier in the semester, for each real vector space $\mathbb{V}$ there is an associated complex vector space $\mathbb{V}_c$ defined as a set to be $\mathbb{V} \times \mathbb{V}$ with elements (v,w) written $v + iw$. The addition is defined component-by-component. And for each complex number $\alpha + i\beta$, $(\alpha + i\beta).(v + iw)$ is defined to be $(\alpha v - \beta w) + i(\beta v + \alpha w)$. The original vector space $\mathbb{V}$ is a subset by $v \mapsto v + i \cdot 0$. And $T : \mathbb{V} \to \mathbb{V}$ extends to a $\mathbb{C}$-linear transformation $T_{\mathbb{C}} : \mathbb{V}_{\mathbb{C}} \to \mathbb{V}_{\mathbb{C}}$ by $T_{\mathbb{C}}\ (v + iw) = T(v) + iT(w)$.

By the Jordan normal form theorem, there exists a direct sum decomposition $\mathbb{V}_{\mathbb{C}} = \mathbb{V}_1, \oplus \ldots\ldots \oplus\ \mathbb{V}_n$ and for each $i = 1,\ldots\ldots,n$ an ordered basis B_i for $\mathbb{V}_i$ s.t.
(1) for each $i = 1,\ldots\ldots,n$, $T\ (\mathbb{V}_i) \subset \mathbb{V}_i$
(2) the corresponding linear transformation $T_i : \mathbb{V}_i \to \mathbb{V}_i$ has matrix

$$[T_i]_{B_i,B_i} = \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \qquad \text{for some } \lambda.$$

For any nonzero $\alpha \in \mathbb{C}$, there is also a basis $B_{i,\alpha}$ s.t.

$$[T_i]_{B_i,\alpha,B_i,\alpha} = [\ ].$$

Indeed, if $B_i = (v_1,\ldots\ldots,v_m)$, then $B_{i,\alpha} = (v_1, \alpha\ v_2, \alpha^2 v_3,\ldots\ldots,\alpha^{m-1}v_m)$.
For each ordered basis B for $\mathbb{V}_i$, there is a "dual basis of coordinates" $x_1,\ldots\ldots,x_m$: $\mathbb{V}_i \to \mathbb{C}$ s.t.
$V = x_1(v)v_1 + \ldots\ldots + x_m(v) \cdot v_m$ for every $v \in \mathbb{V}$: $(B = (v_1,\ldots\ldots,v_m))$. There is a corresponding <u>Hermition inner product</u>,

$$<\cdot,\cdot>_B\ : \mathbb{V}_i \times \mathbb{V}_i \to \mathbb{C}$$

$$<v,w>_B = \sum_{i=1}^{m} x_i(v)\ \overline{x_i(w)}.$$

In particular, this is bilinear, positive definite and $<w,v>_B = (<v,w>_B)$.
And $<T_i v, v>_{Bi\alpha} = \lambda|x_1|^2 + \alpha\ x_1\overline{x_2} + \lambda|x_2|^2 + \alpha\ x_2\overline{x_3} + \ldots\ldots + \alpha x_{m-1}\overline{x_m} + \lambda|x_m|^2$.

<u>Lemma</u>1: For each n, the function on $\mathbb{C}^n$,

$$(x_1,\ldots\ldots,x_n) \mapsto |x_1|^2 - |x_1||x_2| + |x_2|^2 + \ldots + |x_k|^2 - |x_k||x_{k+1}| + |x_{k+1}|^2 + \ldots + -|x_{n-1}||x_n| + |x_n|^2$$

$$= \sum_{k=1}^{n-1} |x_k|^2 - |x_k||x_{k+1}| + |x_n|^2$$

is positive definite.

<u>Proof</u>: it is simply $\frac{1}{2}|x_1|^2 + \frac{1}{2}\sum_{k=1}^{n-1}(|x_k| - |x_{k+1}|)^2 + \frac{1}{2}|x_n|^2$.

Since it is a sum of squares, it is nonnegative. It is zero iff $|x_1| = 0$, $|x_2| - |x_1| = 0, \ldots, |x_n| - |x_{n-1}| = 0$ and $|x_n| = 0$, i.e. $|x_1| = \ldots = |x_n| = 0$.

<u>Lemma 2</u>: If $R_e(\lambda) < 0$ and if $|\alpha| < -R_e(\lambda)$, then $2R_e < T_i v, v >_{\beta i,\alpha}$ is negative definite. Moreover, $2R_e < T_i v, v >_{\beta i,\alpha} \leq 2(R_e(\lambda) + |\alpha|) \cdot (|x_1|^2 + \ldots + |x_n|^2)$.

<u>Proof</u>: $2R_e < T_i v, v >_{\beta i,\alpha} = 2R_e(\lambda)\left(|x_1|^2 + \ldots + |x_n|^2\right) + 2R_e\left(\alpha\ x_1\overline{x_2} + \ldots + \alpha\ x_{n-1}\overline{x_n}\right)$

$\leq 2R_e(\lambda)\left(|x_1|^2 + \ldots + |x_n|^2\right) + 2|\alpha|\left(|x_1|\ |x_2| + \ldots + |x_{n-1}||x_n|\right)$

$= 2(R_e(\lambda) + |\alpha|)\left(|x_1|^2 + \ldots + |x_n|^2\right) - |\alpha|\left(|x_1|^2 - |x_1||x_2| + |x_2|^2 + \ldots + |x_{n-1}|^2 - |x_{n-1}||x_n| + |x_n|^2\right)$

By Lemma 1, $-|\alpha|\left(|x_1|^2 - |x_1||x_2| + |x_2|^2 + \ldots + |x_{n-1}|^2 - |x_{n-1}||x_n| + |x_n|^2\right)$ is negative definite. Because $R_e(\lambda) + \alpha < 0$, also $2(R_e(\lambda) + |\alpha|)\left(|x_1|^2 + \ldots + |x_n|^2\right)$ is negative definite.

For each $i = 1, \ldots, n$, let $|R_e(\lambda)| > \varepsilon_i > 0$. Let $|\alpha_i| + R_e(\lambda) < -\varepsilon_i$, i.e. $0 < |\alpha_i| < |R_e(\lambda)| - \varepsilon_i$.
Define the function $\|\cdot\|^2$ by $\|v_1 + \ldots + v_n\|^2 = \sum_{i=1}^{n} < v_i, v_i >_{Bi,\alpha i}$ where $v_i \in \mathbb{V}_i$

This is a positive definite function. Moreover, $\|\cdot\| := \sqrt{\|\cdot\|^2}$ is a norm. Therefore there is a $\delta > 0$ s.t. if $\|v\| < \delta$, then $\|F(v) - Tv\| \leq \min(\varepsilon_1, \ldots, \varepsilon_n)\|v\|$.
Now $\frac{d}{dt}\|x(t)\|^2 = 2R_e < F(x), x(t) >= \sum_{i=1}^{n} 2R_e < T_i x, x > + 2R_e < F_{(x)} T_v, v >$

So $2R_e < F(x), x > \ \leq \sum_{i=1}^{n} 2R_e < T_i x, x > + \|F_{(x)} - T_x\| \cdot \|x\|$.
By Lemma 2, this is $\leq -2\min(\varepsilon_1, \ldots, \varepsilon_n)\ \|x\|^2 + \|F_{(x)} - T_x\| \cdot \|x\|$
If $\|x\| < \delta$, this is $\leq -2\min(\varepsilon_1, \ldots, \varepsilon_n)\ \|x\|^2 + \min(\varepsilon_1, \ldots, \varepsilon_n)\ \|x\|^2$
$= -\min(\varepsilon_1, \ldots, \varepsilon_n)\ \|x\|^2$

So this is negative semidefinite. Therefore $\|\cdot\|^2$ is a strong Lyapunov function on the ball of radius R centered at p.