18.03 Class 8, Feb 19, 2010

Autonomous equations

[1] Logistic equation
[2] Phase line
[3] Extrema, points of inflection

Announcements:

Final Tuesday, May 18, 9:00 - 12:00, Johnson Track

Hour exam next Wednesday: Rooms to be announced Monday.
Please go to the hour you are registered for. 50 minutes.

[1] Back to qualitative study of differential equations.

I'll use (t,y) today. The general first order equation is

    y'(t) = F(t,y)

Autonomous ODE: y'(t) = g(y) .

Autonomous means conditions are constant in time, though they may depend
on the current value of y .

Eg [Natural growth/decay] Constant growth rate: so y' = k0 y .

Eg Bank account with interest rate NOT depending on time but possibly
depending upon current balance, and constant savings rate: y' = I(y) y + q .

Extended example: Population model with variable growth rate k(y) depending
on the current population but NOT ON TIME; so y' = k(y) y.

Suppose that when y is small the growth rate is approximately k0 , but
that there is a maximal sustainable population p , and as y gets near
to p the growth rate decreases to zero. When y > p , the growth rate
becomes negative; the population declines back to the maximal sustainable
population.

In the simplest version of this, when you graph of   k(y) against y you
get a straight line with vertical intercept k0 and horizontal intercept p :

    k(y) = k0 (1 - (y/p)).

so   k(0) = k0 , and k(p) = 0 . When y > p , k(y) < 0 .

The Logistic Equation is y' = k0 (1 - (y/p)) y = g(y) .

This is more realistic than Natural Growth when you want to account for
limits to growth. It is nonlinear but it IS autonomoous.

Autonomous equations are always separable, but we aim for a qualitative grasp
of solutions. I sketched the nullcline: y = 0 and y = p .
Clearly all the isoclines will be collections of horizontal straight lines
- the differential equation is constant in time.

To get a clear idea of the other isoclines, I will draw a graph of g(y)
as a function of y . It's a parabola opening downward, meeting the horizontal
axis at y = 0 and y = p .

This says that for y < 0   the slopes are negative
               for 0 < y < p the slopes are positive
               for y > p   the slopes are positive.

I drew more isoclines and some solution curves. The bounded solutions
are "logistic curves" or "S-curves." The represent the population drifting
from just above one equilibrium (no oryx) towards the stable population
(p oryx).

If the population exceeds the maximal stable population, it falls back
towards it. The max stable population is a "stable equilibrium;"
the zero population is "unstable."

Values of y for which g(y) = 0 are called "critical points" of
the differential equation y' = g(y) . The are also called "equilibria."

[2] Phase line

Since the direction field is constant horizontally, its essential content
can be conveyed more efficiently. Draw a vertical line. Mark on it the
equilibria, where g(y) = 0 . In between them, draw an upward pointing arrow
if g(y) > 0 and a downward pointing arrow if g(y) < 0 . This simple
diagram tells you roughly how the system behaves. It's called the "phase line."

Question 8.1. In the autonomous equation y' = g(y) , where g(y) has
a graph which I sketch, looking like g(y) = y^3 - y , is the rightmost
critical point

1. Stable
2. Unstable
3. Can't tell, could be either
Blank: don't know

This can be made clear by sketching the phase line. Ans: Unstable.

In terms of the graph of g(y), the stable equilibria occur when
g'(p) < 0 , unstable when g'(p) > 0.

Now, the Kenyan government wants to establish a game preserve on which
it will allow the hunting of oryx. It wants to sell licenses to kill
250 oryx each year and wants to know how big a preserve to establish to
guarantee that this is sustainable. It's known that the population
(measured in kilo-oryx, without hunting) follows the logistic equation

       y' = (a-y)y

Here a is the stable population of oryx, in the absence of hunting.
It is proportional to the area of the preserve. With our 1/4 kilo-oryx
per year hunt, then, we have

    y' = g(y) =(a-y)y - 1/4 = - (y^2 - ay + 1/4)

I invoked the Mathlet <Phase Lines> to visualize what happens.

For small a , the population crashes.

When a is increased to a critical size, a single equilibrium appears,
which is "semistable." You can watch the graph of g(y) at the lower left
of the applet; it's a parabola, opening downward, and it rises as a increases
till at a = 1 it has a single root.

We can solve for where the critical points are, using the quadratic formula:

    y = a/2 +- (1/2)(sqrt(a^2 - 1))

There is just one exactly when the square root is zero, ie a = 1 (or a = -1).

So the Kenyan Chamber of Commerce recommends a preserve of this size, a = 1.

Is this wise? if the population falls every so slightly below the critical
population (which seems to be y = 1/2), it crashes.

When you increase a   further, things get better. There are two critical
points. The farther they are from each other, the more stable the population
system is.

I returned a to zero and opened the Bifurcation Diagram. and watched
what happened as a increased. The bifurcation diagram puts all the
phase lines, for various a , together.

[3] In an autonomous equation y' = g(y) , the conditions represented by
the ODE are constant in time. Direction fields are constant in the horizontal
direction. Consequently, any horizontal (time) translate of a solution is
another solution. A "time translate" of a function y(t) is a function
y(t-t_0): the graph gets shifted horizontally (to the right) by t_0 units.

Ex. Solutions of y' = k_0 y exhibit three different behaviors,
illustrated by

    y = e^{k_0 t} , y = 0 , y = - e^{k_0 t}

Any solution is a horizontal translate of one of them: any solution is either

    y = e^{k_0 (t - t_0)} ,

    y = 0 (whose only time-translate is itself) , or

    y = - e^{{k_0 (t - t_0)}

Question 8.2. Solutions of autonomous equations can have a

strict local maximum.

(A strict local maximum for f(t) is a time t = a such that
f(a) > f(t) for all t near but not equal to a .)

1. True
2. False

Extreme points occur where y' = 0 , i.e. where g(y) = 0. These
are the constant solutions, and they don't have strict maxima or minima.
So it's true, solutions can't have strict extrema.

Question 8.3. Nonconstant solutions of the autonomous ODE y' = g(y)
have inflection points at y for which:

1. g(y)=0
2. g'(y)=0
3. g''(y)=0

y" = g'(y) y'   by the chain rule. So if y" = 0 at y = c then either
g'(c) = 0 (constant solutions) or g'(c) = 0 there. So (2) is the case.
You can see it on the S curve.

18.03 Differential Equations
Spring 2010