

18.03 Class 33, Apr 28, 2010

Eigenvalues and eigenvectors

1. Linear algebra
2. Ray solutions
3. Eigenvalues
4. Eigenvectors
5. Initial values

[1] Prologue on Linear Algebra.

Recall $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$:

A matrix times a column vector is the linear combination of the columns of the matrix weighted by the entries in the column vector.

When is this product zero?

One way is for $x = 0 = y$. If $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ point in different directions, this is the ONLY way. But if they lie along a single line, we can find x and y so that the sum cancels.

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $u = \begin{bmatrix} x \\ y \end{bmatrix}$, so we have been thinking about $Au = 0$ as an equation for u . It always has the "trivial" solution $u = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: 0 is a linear combination of the two columns in a "trivial" way, with 0 coefficients, and we are asking when it is a linear combination of them in a different, "nontrivial" way.

We get a nonzero solution $\begin{bmatrix} x \\ y \end{bmatrix}$ exactly when the slopes of the vectors $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ coincide: $c/a = d/b$, or $ad - bc = 0$. This combination of the entries in A is so important that it's called the "determinant" of the matrix:

$$\det(A) = ad - bc$$

We have found:

Theorem: $Au = 0$ has a nontrivial solution exactly when $\det A = 0$.

If A is a larger *square* matrix the same theorem still holds, with the appropriate definition of the number $\det A$.

[2] Solve $u' = Au$: for example with $A = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$.

The Mathlet "Linear Phase Portraits: Matrix entry" shows that some trajectories seem to be along rays from the origin. (We saw these also in the rabbits example on Monday. They were not present in the Romeo and Juliet example, though!) That is to say, we are going to look for a solution of the form

$$u(t) = r(t) v, \quad v \text{ not } 0$$

One thing for sure: $u'(t)$ also points in the same or reverse direction:

$$u'(t) = r'(t) v$$

Use the equation to express $u'(t)$ in terms of A and v :

$$u'(t) = A u(t) = A r v = r A v$$

$$r' v = r A v$$

So Av points in the same or the reverse direction as v :

$$A v = \lambda v$$

for some number λ . This letter is always used in this context. An *eigenvalue* for A is a number λ such that $A v = \lambda v$ for some nonzero v . A vector v such that $A v = \lambda v$ is an eigenvector for value λ . [Thus the zero vector is an eigenvector for every λ . A number is an eigenvalue for A exactly when it possesses a nonzero eigenvector.]

I showed how this looks on the Mathlet Matrix/Vector.

[3] This is a pure linear algebra problem: A is a square matrix, and we are looking for nonzero vectors v such that $A v = \lambda v$ for some number λ . Let's try to find v . In order to get all the v 's together, right the right hand side as

$$\lambda v = (\lambda I) v$$

where I is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and λI is the matrix with λ down the diagonal. Then we can put this on the left:

$$0 = A v - (\lambda I) v = (A - \lambda I) v$$

Don't forget, we are looking for a nonzero v . We have just found an exact condition for such a solution:

$$\det(A - \lambda I) = 0$$

This is an equation in λ ; we will find λ first, and then set about solving for v (knowing in advance only that there IS a nonzero solution).

In our example, then, we subtract λ from both diagonal entries and then take the determinant:

$$A - \lambda I = \begin{bmatrix} -2 - \lambda & 1 \\ -4 & 3 - \lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (-2-\lambda)(3-\lambda) + 4 \\ &= 1 - \lambda + \lambda^2 - 2 \\ &= \lambda^2 - \lambda - 2 \end{aligned}$$

This is the "characteristic polynomial"

$$p_A(\lambda) = \det(A - \lambda I)$$

of A , and its roots are the "characteristic values" or "eigenvalues" of A .

In our case, $p_A(\lambda) = (\lambda + 1)(\lambda - 2)$

and there are two roots, $\lambda_1 = -1$ and $\lambda_2 = 2$.

(The order is irrelevant.)

[4] Now we can find those special directions. There is one line for λ_1 and another for λ_2 . We have to find nonzero solution v to

$$(A - \lambda I) v = 0$$

eg with $\lambda = \lambda_1 = -1$, $A - \lambda = \begin{bmatrix} -1 & 1 \\ -4 & 4 \end{bmatrix}$

There is a nontrivial linear relation between the columns:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

All we are claiming is that

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and you can check this directly. Any such v (even zero) is called an "eigenvector" of A .

It means that there is a ray solution of the form $r(t) v$ where $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We have

$$r' v = r A v = r \lambda v$$

so (since v is nonzero)

$$r' = \lambda r$$

and solving this goes straight back to Day One:

$$r = c e^{\lambda t}$$

so for us $r = c e^{-t}$ and we have found our first straight line solution:

$$u = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In fact we've found all solutions which occur along that line:

$$u = c e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Any one of these solutions is called a "normal mode."

General fact: the eigenvalue turns out to play a much more important role than it looked like it would: the ray solutions are *exponential* solutions, $e^{\lambda t} v$, where λ is an eigenvalue for

the matrix and v is a nonzero eigenvector for this eigenvalue.

The second eigenvalue, $\lambda_2 = 2$, leads to

$$A - \lambda I = \begin{bmatrix} -4 & 1 \\ -4 & 1 \end{bmatrix}$$

and $\begin{bmatrix} -4 & 1 \\ -4 & 1 \end{bmatrix} v = 0$ has nonzero solution $v = [1; 4]$

so $[1; 4]$ is a nonzero eigenvector for the eigenvalue $\lambda = 2$, and there is another ray solution

$$e^{2t} [1; 4]$$

[5] The general solution to $u' = Au$ will be a linear combination of the two eigensolutions (as long as there are two distinct eigenvalues).

In our example, the general solution is a linear combination of the normal modes:

$$u = c_1 e^{-t} [1; 1] + c_2 e^{2t} [1; 4]$$

We can solve for c_1 and c_2 using an initial condition: say for example

$$u(0) = [1; 0]. \text{ Well,}$$

$$u(0) = c_1 [1; 1] + c_2 [1; 4] = [c_1 + c_2; c_1 + 4c_2]$$

and for this to be $[1; 0]$ we must have $3c_2 = -1 : c_2 = -1/3$;
so $c_1 = -4c_2 = 4/3$:

$$u(t) = (4/3) e^{-t} [1; 1] - (1/3) e^{2t} [1; 4] .$$

When t is very negative, -10 , say, the first term is very big and the second tiny: the solution is very near the line through $[1; -1]$. As t gets near zero, the two terms become comparable and the solution curves around. As t gets large, 10 , say, the second term is very big and the first is tiny: the solution becomes asymptotic to the line through $[1; 1]$.

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