18.03 Hour Exam III Solutions: April 23, 2010

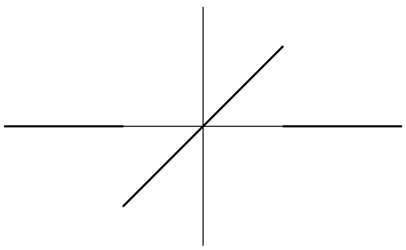
1. (a) The miminal period is 2.

(b) $f(t)$ is even.

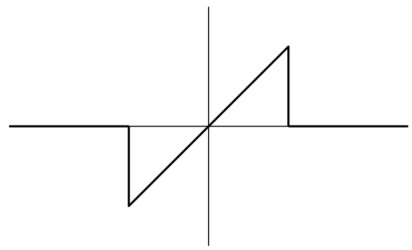
(c) $x_p(t) = \dfrac{1}{\omega_n^2} + \dfrac{\cos(\pi t)}{2(\omega_n^2 - \pi^2)} + \dfrac{\cos(2\pi t)}{4(\omega_n^2 - 4\pi^2)} + \dfrac{\cos(3\pi t)}{8(\omega_n^2 - 9\pi^2)} + \cdots$

(d) There is no periodic solution when $\omega_n = 0, \pi, 2\pi, 3\pi, \ldots$.

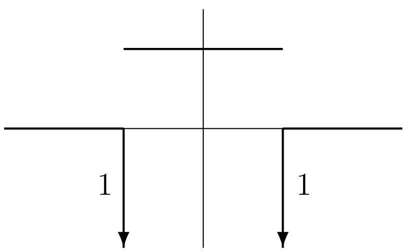
2. (a)



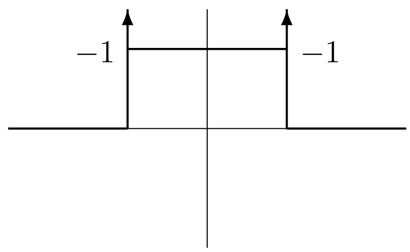
or



(b)



or



(c) $f'(t) = (u(t+1) - u(t-1)) - \delta(t+1) - \delta(t-1)$; $f_r'(t) = u(t+1) - u(t-1)$, $f_s'(t) = -\delta(t+1) - \delta(t-1)$.

3. (a) $v(t) = w(t) * u(t) = \displaystyle\int_0^t w(t-\tau)u(\tau)\,d\tau = \int_0^t (e^{-(t-\tau)} - e^{-3(t-\tau)})\,d\tau$

$= e^{-t}\, e^{\tau}\big|_0^t - e^{-3t}\, \dfrac{e^{3\tau}}{3}\Big|_0^t = (1 - e^{-t}) - \dfrac{1 - e^{-3t}}{3} = \dfrac{2}{3} - e^{-t} + \dfrac{e^{-3t}}{3}.$

(b) $W(s) = \mathcal{L}[w(t)] = \dfrac{1}{s+1} - \dfrac{1}{s+3}$.

(c) $W(s) = \dfrac{(s+3) - (s+1)}{(s+1)(s+3)} = \dfrac{2}{s^2 + 4s + 3}$, so $p(s) = \frac{1}{2}(s^2 + 4s + 3)$.

4. (a) $\dfrac{s-1}{s} = 1 - \dfrac{1}{s} \rightsquigarrow \delta(t) - u(t)$, so $\dfrac{e^{-s}(s-1)}{s} \rightsquigarrow \delta(t-1) - u(t-1)$.

(b) $F(s) = \dfrac{s+10}{s^3 + 2s^2 + 10s} = \dfrac{a}{s} + \dfrac{b(s+1) + c}{(s+1)^2 + 9}$. By coverup, $a = \frac{10}{10} = 1$. By complex coverup (multiply through by $(s+1)^2 + 9$ and set s to be a root, say $-1 + 3i$), $b(3i) + c = \frac{9+3i}{-1+3i} = -3i$, so $b = -1$, $c = 0$, and $F(s) = \dfrac{1}{s} - \dfrac{s+1}{(s+1)^2 + 9}$, which is the Laplace transform of $1 - e^{-t}\cos(3t)$.

5. (a) $\{0, -1 + 3i, -1 - 3i\}$.

(b) $X(s) = W(s)F(s)$. $F(s) = \dfrac{2}{s^2 + 4}$, so $X(s) = \left(\dfrac{s+10}{s^3 + 2s^2 + 10s}\right)\left(\dfrac{2}{s^2 + 4}\right)$.