18.02 Problem Set 9, Part II Solutions

1. (a) If C is a simple closed curve enclosing the region R then

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_R \text{curl } \mathbf{F} \; dx \; dy \\
&= \int\int_R (6x - x^3)_x - (y^3 - 6y)_y \; dx \; dy \\
&= \int\int_R (6 - 3x^2 + 6 - 3y^2) \; dx \; dy \\
&= \int\int_R (12 - 3x^2 - 3y^2) \; dx \; dy
\end{aligned}$$

We seek to maximize this integral. The function $12 - 3x^2 - 3y^2$ is ≥ 0 when

$$3x^2 + 3y^2 \leq 12$$

or $x^2 + y^2 \leq 2^2$. So the function is ≥ 0 on the disc D of radius 2 centered at 0. When $R = D$ we maximize this integral. Thus when C is the curve tracing the boundary of D in the counter-clockwise direction, we maximize $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

(b) We just calculate

$$\begin{aligned}
\int\int_D (12 - 3x^3 - 3y^2) \; dx \; dy &= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (12 - 3r^2) \; rdr \; d\theta \\
&= \int_{\theta=0}^{2\pi} \left[6r^2 - \frac{3}{4} r^4 \right]_0^2 \; d\theta \\
&= \int_{\theta=0}^{2\pi} 6 \cdot 2^2 - \frac{3}{4} 2^4 \; d\theta \\
&= 2\pi(24 - 12) = 24\pi
\end{aligned}$$

2. (a) The equation of continuity as stated is equivalent to the the statement that $\displaystyle\iint_{\mathcal{R}} \frac{\partial \rho}{\partial t} \, dA + \iint_{\mathcal{R}} \text{div}(\mathbf{F}) \, \text{dA} = 0$ for all simple bounded regions $\mathcal{R}$. The first integral in the sum is equal to $\frac{d}{dt} M(\mathcal{R}; t)$, where $M(\mathcal{R}; t) = \iint_{\mathcal{R}} \rho(x, y, t) \, dA$ is the mass contained in the region $\mathcal{R}$ at time t. By Green's theorem, the second (or divergence) integral is equal to $\displaystyle\oint_C \mathbf{F}(x, y, t) \cdot \hat{\mathbf{n}}_{\text{out}} \; ds$, which is the mass flux *out* of the region $\mathcal{R}$ at time t, that is, the net rate at which mass is leaving $\mathcal{R}$ through the boundary C, in mass per unit time. Thus mass is conserved if and only if this net boundary rate, which is equal to the divergence integral, is equal to $-\frac{d}{dt}M(\mathcal{R},t)$. (To check that the signs are right, take for example $\frac{d}{dt}M(\mathcal{R},t) > 0$; then the mass in $\mathcal{R}$ is increasing, and so mass must be coming *into* $\mathcal{R}$ through C at that rate.)

(b) $\operatorname{div}(\mathrm{g}\,\mathbf{G}) = \frac{\partial(gM)}{\partial x} + \frac{\partial(gM)}{\partial y} = (g_x M + gM_x) + (g_y N + gN_y) =$
$= (g_x M + g_y N) + (gM_x + gN_y) = \mathbf{G}\cdot\nabla g + g\operatorname{div}(\mathbf{G})$.

(c) $\dfrac{\partial\rho}{\partial t} + \operatorname{div}(\mathbf{F}) = \dfrac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\operatorname{div}(\mathbf{v}) = \dfrac{D\rho}{Dt} + \rho\operatorname{div}(\mathbf{v})$, with the first equality by part(b) and the second by the general chain rule result for convective derivatives (p-set 5, #2). Thus the equation of continuity defined as $\frac{\partial\rho}{\partial t} + \operatorname{div}(\mathbf{F}) = 0$ holds if and only if $\frac{D\rho}{Dt} + \rho\operatorname{div}(\mathbf{v}) = 0$, from which it follows that $\dfrac{D\rho}{Dt} = 0$ if and only if $\operatorname{div}(\mathbf{v}) = 0$.

3. **(i)**. Circular flow rotating around the origin O, speeding up with time. $\frac{\partial\rho}{\partial t} = 0$, $\mathbf{v}\cdot\nabla\rho = 0$ and $\operatorname{div}(\mathbf{v}) = 0$, for all (x,y,t), so by 4(c) above the eqn. of continuity is satisfied. $\operatorname{div}(\mathbf{v}) = 0$, so the flow is incompressible; and since flow is not homogeneous (i.e. the density is not constant), it is stratified. (Even though the flow is not steady, we do have $\rho = \rho(x,y)$ only, and so incompressibility implies that $\mathbf{v}\cdot\nabla\rho = 0$, as in p-set 5 #3(b); in this case this is also clear, since the gradients of the density $\nabla\rho = \frac{1}{r}\langle x,y\rangle$ are radial.)

(ii). The flow paths are hyperbolas (as in p-set 7 #5 case C). The flow is slowing down with time. Again by direct computation we see that $\frac{\partial\rho}{\partial t} = 0$, $\mathbf{v}\cdot\nabla\rho = 0$ and $\operatorname{div}(\mathbf{v}) = 0$, for all (x,y,t), so the equation of continuity is satisfied; $\operatorname{div}(\mathbf{v}) = 0$ gives that flow is incompressible; and since flow is not homogeneous, it is stratified, again with $\rho = \rho(x,y)$ only, and $\mathbf{v}\cdot\nabla\rho = 0$.

(iii). The flow is radial outward from the origin. The flow paths are half-rays, i.e. straight lines starting from O. The flow is speeding up with time. We compute $\frac{\partial\rho}{\partial t} = -2\,t\,\mathrm{e}^{-t^2}$, and $\operatorname{div}(\rho(\mathrm{t})\mathbf{v}) = \rho(t)\operatorname{div}(\mathbf{v}) = \mathrm{e}^{-t^2}\,2\,t$, so the equation of continuity is satisfied. However $\operatorname{div}(\mathbf{v}) = 2\,\mathrm{t} \neq 0$, so the flow is **not** incompressible.

Additional material (*optional* - for those who are interested in the completion of this the story): we need to show, as promised in p-set 7, that 'volume-incompressibility,' as defined in p-set 7 #5, is equivalent to the original definition of incompressibility as $\frac{D\rho}{Dt} = 0$. This now goes via the equivalent condition $\operatorname{div}(\mathbf{v}) = 0$ as follows. First, the chain rule is used to prove that if $|J(x,y,z,t)|$ is the Jacobian determinant of the flow map $\varphi(x,y,z,t)$ (in the general 3D case), then $|J|$ satisfies the equation

$$\frac{\partial |J|}{\partial t} = |J| \operatorname{div}(\mathbf{v}).$$

(This takes a bit of work, but it's true.)
Thus $|J(x, y, z, t)|$ is constant in t if and only if $\operatorname{div}(\mathbf{v}) = 0$, i.e. if and only if the flow is incompressible.
To show that this constant is equal to 1 for all (x, y, z), we combine the equation $|J(x, y, z, t)| = |J(x, y, z, 0)|$ for all t (i.e. $|J(x, y, z, t)|$ is constant in t) with the equation $|J(x, y, z, 0)| = 1$ for all (x, y, z). To see the second equation, note that by definition the flow map $\varphi(x, y, z, 0) = (x, y, z)$ is the identity map at $t = 0$, and also that the Jacobian of the identity map is identically equal to 1. This shows that a flow is incompressible if and only if $|J(x, y, z, t)| = 1$ for all (x, y, z, t), which is the condition for volume-incompressibility.