# 18.02 Problem Set 10 - Solutions of Part B

**Problem 1**

a) Green's theorem in normal form: $\iint_R (M_x + N_y)\mathrm{dA} = \int_C -N\mathrm{d}x + M\mathrm{d}y$.
For $M = 0$ and $N = y^{2n-1}$ applied to the unit circle, we have

$$\int_0^{2\pi}\int_0^1 (2n-1)y^{2n-2}\ r\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} -y^{2n-1}\mathrm{d}x$$

$$\int_0^{2\pi}\int_0^1 (2n-1)r^{2n-1}\sin^{2n-2}\theta\ \mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\sin^{2n}\theta\ \mathrm{d}\theta$$

$$\frac{2n-1}{2n}\int_0^{2\pi}\sin^{2n-2}\theta\ \mathrm{d}\theta = \int_0^{2\pi}\sin^{2n}\theta\ \mathrm{d}\theta$$

that is, $A_n = \dfrac{2n-1}{2n}A_{n-1}$.

b) $A_0 = \displaystyle\int_0^{2\pi}\mathrm{d}\theta = 2\pi$.

$$A_n = \frac{2n-1}{2n}A_{n-1} = \frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}A_0 = \frac{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}2\pi.$$

c) On the $x$-axis $\vec{\mathbf{G}} = 0$, so it is sufficient to compute the line integral on the unit upper half-circle.

$$\int_0^{\pi}\int_0^1 2n\, y^{2n-1}\ r\mathrm{d}r\mathrm{d}\theta = \int_0^{\pi} -y^{2n}\mathrm{d}x$$

$$2n\int_0^{\pi}\int_0^1 r^{2n}\sin^{2n-1}\theta\ \mathrm{d}r\mathrm{d}\theta = \int_0^{\pi}\sin^{2n+1}\theta\ \mathrm{d}\theta$$

$$\frac{2n}{2n+1}\int_0^{\pi}\sin^{2n-1}\theta\ \mathrm{d}\theta = \int_0^{\pi}\sin^{2n+1}\theta\ \mathrm{d}\theta$$

that is, $B_n = \dfrac{2n}{2n+1}B_{n-1}$.

d) $B_0 = \displaystyle\int_0^{\pi}\sin\theta\ \mathrm{d}\theta = [-\cos\theta]_0^{\pi} = 2$.

$$B_n = \frac{2n}{2n+1}B_{n-1} = \frac{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}{3\cdot 5\cdot 7\cdots(2n-1)\cdot(2n+1)}B_0 = \frac{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)}{3\cdot 5\cdot 7\cdots(2n-1)\cdot(2n+1)}2.$$

Integrating $\sin^{2n+1}\theta$ over $[0, 2\pi]$ would be useless, because $\displaystyle\int_0^{2\pi}\sin^{2n+1}\theta\ \mathrm{d}\theta = 0$.

In fact, it is equal to $\int_{-\pi}^{\pi} \sin^{2n+1}\theta \; \mathrm{d}\theta = \int_{-\pi}^{0} \sin^{2n+1}\theta \; \mathrm{d}\theta + \int_{0}^{\pi} \sin^{2n+1}\theta \; \mathrm{d}\theta$ and $\int_{-\pi}^{0} \sin^{2n+1}\theta \; \mathrm{d}\theta = -\int_{0}^{\pi} \sin^{2n+1}\theta \; \mathrm{d}\theta$, because $\sin^{2n+1}(-\theta) = -\sin^{2n+1}\theta$.

**Problem 2**

a) About 30%.

Assume the Earth spherical with radius $R$.
Then its surface has Total Area $= 4\pi R^2$.
The region $R$ South of Rio de Janeiro has $90^o + 23^o = 113^o < \varphi \leq 180^o$, so it has $\text{Area}(R) = \int_0^{2\pi} \int_{113\pi/180}^{\pi} R^2 \sin\varphi \; \mathrm{d}\varphi \mathrm{d}\theta = 2\pi R^2 \left[1 + \cos\left(\frac{113}{180}\pi\right)\right]$.
Hence, the percentage is

$$100 \frac{\text{Area}(R)}{\text{Total Area}} \% = \frac{100}{2} \left[1 + \cos\left(\frac{113}{180}\pi\right)\right] \% \approx 50(1 - 0.391)\% \approx 30\%.$$

b) The average latitude is approximately $32^o 42'$.
Hence, Boston ($42^o$) is North of the average.
For example, Tripoli (Lybia) is at $32^o 48'$, so within one degree from the average.

The surface area of the Northern hemisphere is $2\pi R^2$.
The average $\varphi$ is $\overline{\varphi} = \frac{1}{2\pi R^2} \int_0^{2\pi} \int_0^{\pi/2} \varphi R^2 \sin\varphi \; \mathrm{d}\varphi \mathrm{d}\theta = \int_0^{\pi/2} \varphi \sin\varphi \; \mathrm{d}\varphi =$
$= [-\varphi \cos\varphi]_0^{\pi/2} - \int_0^{\pi/2} (-\cos\varphi) \; \mathrm{d}\varphi = [\sin\varphi]_0^{\pi/2} = 1$ radiant $\approx 57^o 18'$.
Hence, the average latitude is approximately $90^o - 57^o 18' = 32^o 42'$.

**Problem 3**

The total flux out of the cylinder is $3\pi a^2$.

We use the definition of flux $\iint_R \vec{\mathbf{F}} \cdot \hat{n}\ \mathrm{dS}$.

**(bottom)** On the bottom face ($z = 0$) we have: $\hat{n} = -\hat{\boldsymbol{k}}$ and $\vec{\mathbf{F}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, so $\vec{\mathbf{F}} \cdot \hat{n} = 0$. Hence the flux is zero.

**(top)** On the top face ($z = 1$) we have: $\hat{n} = \hat{\boldsymbol{k}}$ and $\vec{\mathbf{F}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$, so $\vec{\mathbf{F}} \cdot \hat{n} = 1$. Hence (Flux) $= \iint_{\text{top}} 1\ \mathrm{dS} = (\text{Area}) = \pi a^2$.

**(lateral)** On the lateral surface ($x^2 + y^2 = a^2$) we have: $\hat{n} = \dfrac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{a}$, so $\vec{\mathbf{F}} \cdot \hat{n} = \dfrac{x^2 + y^2}{a} = a$. Hence (Flux) $= \iint_{\text{lateral}} a\ \mathrm{dS} = a \cdot (\text{Area}) = a \cdot (2\pi a \cdot 1) = 2\pi a^2$.

Finally, the total flux is $0 + \pi a^2 + 2\pi a^2 = 3\pi a^2$.