18.02 Problem Set 3 - Solutions of Part B

Problem 1

The partial derivatives at the point are approximately $f_x \approx -4/5$ and $f_y \approx -1/5$.
The direction of steepest ascent is approximately at 104 degrees from North.

The derivative $f_x \approx \dfrac{\Delta z}{\Delta x}$, where Δx is the horizontal distance between the level curve where the point lies and the next one, and Δz is 100 or -100 depending on whether we next level curve is at higher or lower altitude.
In our case, the horizontal distance on our map between the level line we are interested in an the next one is approximately 1/16 inches, so $\Delta x = 1/16 \cdot 2000 = 125$ feet. As the altitude decreases, $\Delta z = -100$. Hence we get $f_x \approx \dfrac{-100}{125} = -\dfrac{4}{5}$. Similarly, for f_y we have $\Delta y = 1/4 \cdot 2000 = 500$ and $\Delta z = -100$, which gives us $f_y \approx \dfrac{-100}{500} = -\dfrac{1}{5}$.

The direction of steepest ascent is the same as $\langle f_x, f_y \rangle$. Hence it is at

$$\alpha = 90 + \arctan\left(\frac{-1/5}{-4/5}\right) = 90 + \arctan\left(\frac{1}{4}\right) \approx 90 + 14 = 104$$

degrees from North.

Problem 2

a) $\mathbf{x}^{(1)} = \begin{pmatrix} 15.28 \\ 9.56 \\ 5.99 \\ 22.77 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 12.31 \\ 11.67 \\ 6.78 \\ 22.84 \end{pmatrix}, \quad \mathbf{x}^{(8)} = \begin{pmatrix} 10.36 \\ 12.79 \\ 6.70 \\ 23.75 \end{pmatrix}.$

In fact it is sufficient to enter:
```
M = [.50, .10, .05, .15; .20, .60, .10, .10; .15, .10, .40, .05; .15, .20, .45, .70]
b = [22, 4, 2.6, 25]'
x1 = M * b
x2 = (M^2) * b
x8 = (M^8) * b
```
where we used the symbol `x1` for $\mathbf{x}^{(1)}$, the symbol `x2` for $\mathbf{x}^{(2)}$ and the symbol `x8` for $\mathbf{x}^{(8)}$.

b) The final market shares are approximately: AOL 10.35 million subscribers, Earthlink 12.79 million, MSN 6.70 million and Other.com 23.75 million (the total does not sum up to 53.6 million just because of the rounding).
These shares (± 100000 subscribers) first appear after 6 years (when we get: AOL 10.3795 million subscribers, Earthlink 12.7848 million, MSN 6.7280 million and Other.com 23.7076 million).

Just enter
`(M^1000) * b`
(vector of subscribers after 1000 years!).
To discover when these shares (± 100000 subscribers) first appear, just try.

c) The final market shares are the same as in part b). In fact they only depend on the matrix `M` and on the total number of subscribers.

d) The entry $m_{i,j}$ of M at the row i column j is equal to

$$m_{i,j} = \frac{\text{subscribers of } j \text{ at the year } N \text{ who subscribe to } i \text{ at the year } N+1}{\text{subscribers of } j \text{ at the year } N}$$

so we immediately get that the sum of entries of the column j is the ratio between (subscribers of j at the year N who subscribe to something at the year $N+1$) and (subscribers of j at the year N). As we are assuming that the total number of subscribers stays the same every year, then this ratio is clearly 1.

e) It is sufficient to add an entry $x_5 = \{\text{nonsubscribers}\}$. Consequently, the matrix M will be a (5×5)-matrix: the new entries will take care of those people who do not subscribe at some year.

Problem 3

a) $\Delta r \approx \left(-\frac{1}{2} + \frac{b}{2\sqrt{b^2-4c}}\right)\Delta b + \left(-\frac{1}{\sqrt{b^2-4c}}\right)\Delta c.$
In our particular case, $\Delta r \approx 0.06$.

In fact, the largest root is $r(b,c) = \dfrac{-b+\sqrt{b^2-4c}}{2}$.
Using the first order approximation $\Delta r \approx \dfrac{\partial r}{\partial b}\Delta b + \dfrac{\partial r}{\partial c}\Delta c$, we get

$$\Delta r \approx \left(-\frac{1}{2} + \frac{b}{2\sqrt{b^2-4c}}\right)\Delta b + \left(-\frac{1}{\sqrt{b^2-4c}}\right)\Delta c.$$

In our case, $b = -7$, $c = 12$, $\Delta b = -0.01$ and $\Delta c = -0.02$. Hence

$$\begin{aligned}\Delta r \approx & \left(-\frac{1}{2} + \frac{-7}{2\sqrt{49-48}}\right)(-0.01) + \left(-\frac{1}{\sqrt{49-48}}\right)(-0.02) = \\ = & (-4)(-0.01) + (-1)(-0.02) = 0.06.\end{aligned}$$

b) $r(b,c)$ is more sensitive to small changes of b.

In fact, from (a) we have $\dfrac{\partial r}{\partial b}(-7,12) = -4$ and $\dfrac{\partial r}{\partial c}(-7,12) = -1$.
Because $\left|\dfrac{\partial r}{\partial b}(-7,12)\right| > \left|\dfrac{\partial r}{\partial c}(-7,12)\right|$, the larger root r is more sensitive to small changes of b.

Problem 4

a) $A = \begin{pmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & n \end{pmatrix}$, $z = \begin{pmatrix} a \\ b \end{pmatrix}$ and $r = \begin{pmatrix} \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$.

```
u = ones(1,10)
r = [x*y',u*y']'
A = [x*x',x*u';x*u',u*u']
z = A^(-1)*r
```

c) The coefficients are $a = 133.4397$, $b = -10557$.
The worst error (actual value minus predicted value) is 460.1853.

Enter:

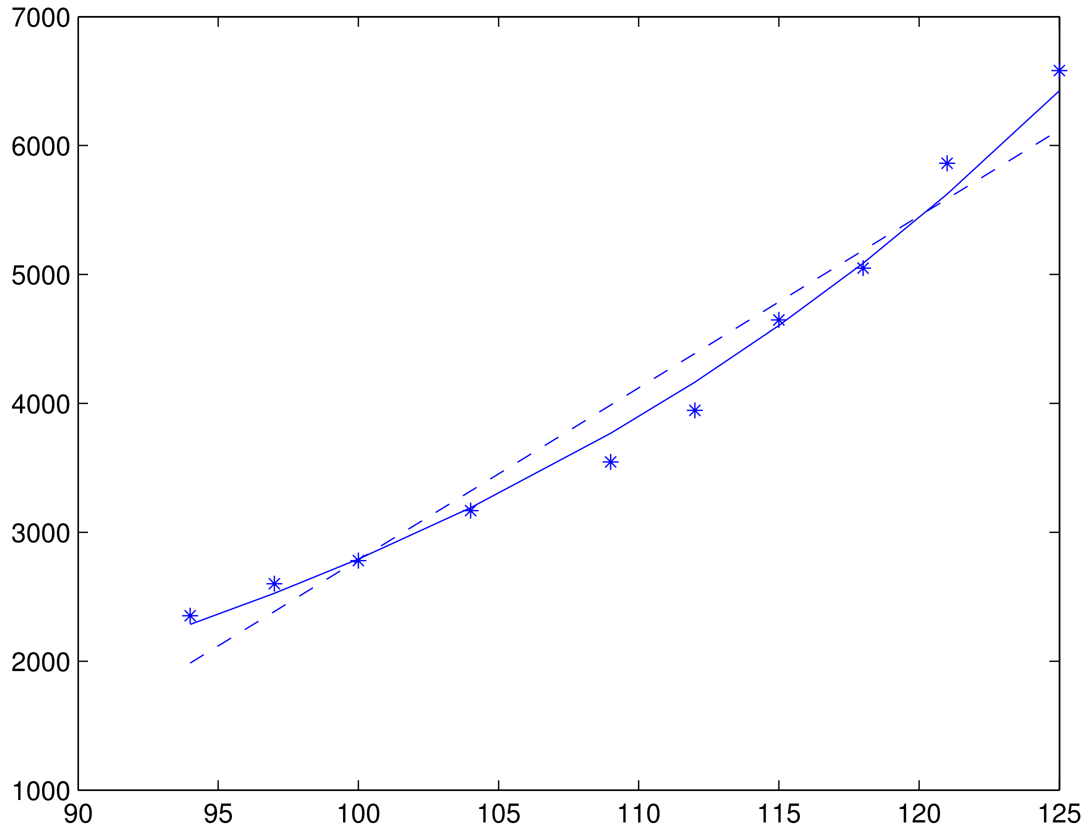
```
a = [1,0]*z
b = [0,1]*z
e = y - a*x + b*u
```

Then look for the largest entry (in absolute value) of the vector `e`.

d) $a_1 = 0.0333$, $b_1 = 4.6$
The worst error is 240.5397.

Enter:
`y1 = log(y)`
`r1 = [x * y1', u * y1']'`
`z1 = A^(-1) * r1`
`a1 = [1, 0] * z1`
`b1 = [0, 1] * z1`
`e1 = y1 - exp(a1 * x + b1 * u)`
Look for the largest entry (in absolute value) of the vector `e1`.

e)



The graph is obtained by entering (starting with a non-hold mode):
`plot(x, y, '*')`
`hold`
`plot(x, a * x + b * u, '--')`

`plot(x, exp(a1 * x + b1 * u))`

f) On Jan 1, the esponential law gives 103 cases.
On Jul 1, 43003 cases.
On Dec 1, 7067600 cases.
(Notice that all answers are rounded.)

Jan 1 is the first day of the year 2003, Jul 1 is the 182nd and Dec 1 is the 335th.
Then it is sufficient to enter:

```
exp(a1 * 1 + b1)
exp(a1 * 182 + b1)
exp(a1 * 335 + b1)
```

to get the result.