18.02 Problem Set 7 - Solutions of Part B

Problem 1

a) $f(r) = \begin{cases} \dfrac{r}{50} & \text{for } 0 \le r \le 10 \\ 0 & \text{otherwise} \end{cases}.$

For $0 \le a \le b \le 10$, $P(a \le r \le b) = \dfrac{1}{\pi(10)^2} \displaystyle\int_0^{2\pi} \int_a^b r \mathrm{d}r \mathrm{d}\theta =$

$= \dfrac{1}{100\pi} \left(\displaystyle\int_0^{2\pi} \mathrm{d}\theta \right) \left(\displaystyle\int_a^b r \mathrm{d}r \right) = \displaystyle\int_a^b \dfrac{r}{50} \mathrm{d}r.$

Therefore $f(r) = \dfrac{r}{50}$ for $0 \le r \le 10$.

b) $f(r) = \begin{cases} 4r^3 & \text{for } 0 \le r \le 1 \\ 0 & \text{otherwise} \end{cases}.$

For $0 \le a \le b \le 1$, $P(a \le r \le b) = \dfrac{1}{M} \displaystyle\int_0^{\pi} \int_a^b y^2 r \mathrm{d}r \mathrm{d}\theta =$

$= \dfrac{1}{M} \displaystyle\int_0^{\pi} \int_a^b r^3 \sin^2\theta \mathrm{d}r \mathrm{d}\theta = \dfrac{1}{M} \displaystyle\int_0^{\pi} \sin^2\theta \mathrm{d}\theta \int_a^b r^3 \mathrm{d}r = \int_a^b 4r^3 \mathrm{d}r$

because $M = \dfrac{\pi}{8}$ from *PS6 - Problem 4* and $\displaystyle\int_0^{\pi} \sin^2\theta \mathrm{d}\theta = 2 \int_0^{\pi/2} \sin^2\theta \mathrm{d}\theta = \dfrac{\pi}{2}$

(from *Notes - Table 3B* or using $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$).

Hence, $f(r) = 4r^3$ for $0 \le r \le 1$.

Problem 2

a) The average distance is $\overline{d} = \dfrac{32}{9\pi} a$.

In Cartesian coordinates $\overline{d} = \dfrac{1}{\pi a^2} \displaystyle\int_0^{2a} \int_{-\sqrt{2ax - x^2}}^{\sqrt{2ax - x^2}} \sqrt{x^2 + y^2} \mathrm{d}x \mathrm{d}y$

(or $\dfrac{2}{\pi a^2} \displaystyle\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} \sqrt{x^2 + y^2} \mathrm{d}x \mathrm{d}y$).

In polar coordinates $\overline{d} = \dfrac{1}{\pi a^2} \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r{\cdot}r\mathrm{d}r\mathrm{d}\theta = \dfrac{2}{\pi a^2} \displaystyle\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2\mathrm{d}r\mathrm{d}\theta =$

$= \dfrac{2}{\pi a^2} \displaystyle\int_0^{\frac{\pi}{2}} \dfrac{(2a\cos\theta)^3}{3}\mathrm{d}\theta = \dfrac{16a}{3\pi} \displaystyle\int_0^{\frac{\pi}{2}} \cos^3\theta\mathrm{d}\theta = \dfrac{32a}{9\pi}$

because $\displaystyle\int_0^{\pi/2} \cos^3\theta\mathrm{d}\theta = \dfrac{2}{3}$ (using *Notes - Table 3B*
or $\cos^3\theta = (1-\sin^2\theta)\cos\theta$).
Therefore $\overline{d} = \dfrac{32}{9\pi}a$.

b) The probability density is $f(r) = \begin{cases} \dfrac{2r}{\pi a^2}\cos^{-1}\left(\dfrac{r}{2a}\right) & \text{for } 0 \le r \le 2a \\ 0 & \text{otherwise} \end{cases}$.

For $0 \le b \le c \le 2a$, exchange the order of integration:
$r \le 2a\cos\theta \quad \Leftrightarrow \quad \dfrac{r}{2a} \le \cos\theta \quad \Leftrightarrow \quad -\cos^{-1}\left(\dfrac{r}{2a}\right) \le \theta \le \cos^{-1}\left(\dfrac{r}{2a}\right)$.
$P(b \le r \le c) = \dfrac{1}{\pi a^2}\displaystyle\int_b^c \int_{-\cos^{-1}(r/2a)}^{\cos^{-1}(r/2a)} r\mathrm{d}\theta\mathrm{d}r = \int_b^c \dfrac{2r}{\pi a^2}\cos^{-1}\left(\dfrac{r}{2a}\right)\mathrm{d}r$.
Hence, $f(r) = \dfrac{2r}{\pi a^2}\cos^{-1}\left(\dfrac{r}{2a}\right)$ for $0 \le r \le 2a$.

c) The probability density is $\begin{cases} g(s) = \dfrac{2s^{-1/3}}{3\pi a^2}\cos^{-1}\left(\dfrac{s^{1/3}}{2a}\right) & \text{for } 0 \le s \le 8a^3 \\ 0 & \text{otherwise} \end{cases}$.

Let $s = r^3$ and $g(s)$ be the probability density of r^3.
For $0 \le b \le c \le (2a)^3 = 8a^3$, $P(b \le s \le c) = P(b^{1/3} \le r \le c^{1/3}) =$
$= \displaystyle\int_{b^{1/3}}^{c^{1/3}} f(r)\mathrm{d}r = \int_b^c f(s^{1/3})\dfrac{s^{-2/3}}{3}\mathrm{d}s$
where $f(r)$ is the probability density from (b), because $r = s^{1/3}$ and
$\mathrm{d}r = \dfrac{s^{-2/3}}{3}\mathrm{d}s$.
Hence, $g(s) = f(s^{1/3})\dfrac{s^{-2/3}}{3} = \dfrac{2s^{-1/3}}{3\pi a^2}\cos^{-1}\left(\dfrac{s^{1/3}}{2a}\right)$ for $0 \le s \le 8a^3$.

Problem 3

a) The average is $\overline{x} = \frac{1}{4}$.

The probability density for x is $f(x) = \begin{cases} 3(x-1)^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

The tetrahedron is given by $\begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \\ y \le z \le 1 \end{cases}$.

The volume of the tetrahedron is $V = \int_0^1 \int_x^1 \int_y^1 \mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^1 \int_x^1 (1-y)\mathrm{d}y\mathrm{d}x =$
$= \int_0^1 \left[y - \frac{y^2}{2}\right]_{y=x}^{y=1} \mathrm{d}x = \int_0^1 \left(1 - \frac{1}{2} - x + \frac{x^2}{2}\right) \mathrm{d}x = \int_0^1 \frac{(x-1)^2}{2} \mathrm{d}x =$
$= \left[\frac{(x-1)^3}{6}\right]_{x=0}^{x=1} = \frac{1}{6}$.
The average is $\overline{x} = \frac{1}{V} \int_0^1 \int_x^1 \int_y^1 x\mathrm{d}z\mathrm{d}y\mathrm{d}x = 6 \int_0^1 \int_x^1 x(1-y)\mathrm{d}y\mathrm{d}x =$
$= 6 \int_0^1 \frac{x - 2x^2 + x^3}{2} \mathrm{d}x = 3 \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{x=0}^{x=1} = 3\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) =$
$= 3\frac{6-8+3}{12} = \frac{1}{4}$.
(Any order of integration, such as $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ can be used, but in the next part we want to integrate in y and z before x.)
The probability of x being between $0 \le a$ and $b \le 1$ is
$P(a < x < b) = \int_a^b f(x)\mathrm{d}x == \frac{1}{V} \int_a^b \int_x^1 \int_y^1 \mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_a^b \int_x^1 6(1-y)\mathrm{d}y\mathrm{d}x =$
$= \int_a^b 6\frac{(1-x)^2}{2}\mathrm{d}x = \int_a^b 3(1-x)^2\mathrm{d}x.$
Hence, $f(x) = 3(1-x)^2$ for $0 \le x \le 1$ and $f(x) = 0$ outside this interval.

b) $P(x < \frac{1}{2}) = \frac{7}{8}$ and $P(x < \frac{1}{4}) = \frac{37}{64}$.

From (a), we have
$P(x < \frac{1}{2}) = \int_0^{\frac{1}{2}} f(x)\mathrm{d}x = \int_0^{\frac{1}{2}} 3(1-x)^2\mathrm{d}x = \left[(x-1)^3\right]_0^{\frac{1}{2}} = -\frac{1}{8} + 1 = \frac{7}{8}$.
$P(x < \frac{1}{4}) = \int_0^{\frac{1}{4}} f(x)\mathrm{d}x = \int_0^{\frac{1}{4}} 3(1-x)^2\mathrm{d}x = \left[(x-1)^3\right]_0^{\frac{1}{4}} = -\frac{27}{64} + 1 = \frac{37}{64}$.

c) $\overline{x}_1 = \dfrac{1}{n+1}$.
(Extra-credit, added at the end of the term separately from all other scores.)

In fact $\overline{x}_1 = \dfrac{\int_R x_1 \mathrm{d}x_1 \cdots \mathrm{d}x_n}{\int_R \mathrm{d}x_1 \cdots \mathrm{d}x_n}$, where R is given by $\begin{cases} 0 \leq x_n \leq 1 \\ 0 \leq x_{n-1} \leq x_n \\ \cdots \\ 0 \leq x_1 \leq x_2 \end{cases}$.

The denominator is $\int_0^1 \int_0^{x_{n-1}} \cdots \int_0^{x_2} dx_1 \cdots \mathrm{d}x_{n-1}\mathrm{d}x_n =$
$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} x_2 \mathrm{d}x_2 \cdots \mathrm{d}x_n = \int_0^1 \int_0^{x_n} \cdots \int_0^{x_4} \frac{x_3^2}{2} \mathrm{d}x_3 \cdots \mathrm{d}x_n =$
$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_5} \frac{x_4^3}{2 \cdot 3} \mathrm{d}x_4 \cdots dx_n = \cdots = \left[\frac{x_n^n}{2 \cdot 3 \cdots n} \right]_0^1 = \frac{1}{2 \cdot 3 \cdots n}$.
The numerator is $\int_0^1 \int_0^{x_n} \cdots \int_0^{x_2} x_1 dx_1 \cdots \mathrm{d}x_{n-1}\mathrm{d}x_n =$
$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_3} \frac{x_2^2}{2} \mathrm{d}x_2 \cdots \mathrm{d}x_n = \int_0^1 \int_0^{x_n} \cdots \int_0^{x_4} \frac{x_3^3}{2 \cdot 3} \mathrm{d}x_3 \cdots \mathrm{d}x_n =$
$= \int_0^1 \int_0^{x_n} \cdots \int_0^{x_5} \frac{x_4^4}{2 \cdot 3 \cdot 4} \mathrm{d}x_4 \cdots dx_n = \cdots = \left[\frac{x_n^n}{2 \cdot 3 \cdots n(n+1)} \right]_0^1 = \frac{1}{2 \cdot 3 \cdots n(n+1)}$.
Hence $\overline{x}_1 = \dfrac{2 \cdot 3 \cdots n}{2 \cdot 3 \cdots n(n+1)} = \dfrac{1}{n+1}$.

Problem 4

a) $\int_0^1 \int_0^1 \mathrm{d}x\mathrm{d}y = \int_0^1 \int_{2u-1}^{2u} \frac{1}{2\sqrt{2u-v}} \mathrm{d}v\mathrm{d}u = 1.$

The region of integration is a square in the xy-plane with vertices $P = (0,0)$, $Q = (1,0)$, $R = (1,1)$ and $S = (0,1)$.
In the uv-plane the same region becomes a parallelogram with vertices $P = (0,0)$, $Q = (1,2)$, $R = (1,1)$ and $S = (0,-1)$.
In fact, $y^2 = 2x - v = 2x - u$ and $0 \leq y^2 \leq 1$ give $0 \leq 2u - v \leq 1$, so that the region of integration is described (in terms of u and v) as $\begin{cases} 0 \leq u \leq 1 \\ 2u - 1 \leq v \leq 2u \end{cases}$.
Moreover the chain rule tells us that

$$\begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -2y \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

so that $\quad \mathrm{d}u\mathrm{d}v = \left|\det\begin{pmatrix} 1 & 0 \\ 2 & -2y \end{pmatrix}\right| \mathrm{d}x\mathrm{d}y = |-2y|\mathrm{d}x\mathrm{d}y = 2y\mathrm{d}x\mathrm{d}y$
(where the last equality holds because y is positive on our domain of integration).
As $y = \sqrt{2u - v}$ on the domain of integration, we can also write
$\mathrm{d}x\mathrm{d}y = \dfrac{\mathrm{d}u\mathrm{d}v}{2\sqrt{2u-v}}$.
Hence $\displaystyle\int_0^1\int_0^1 \mathrm{d}x\mathrm{d}y = \int_0^1\int_{2u-1}^{2u} \frac{1}{2\sqrt{2u-v}}\mathrm{d}v\mathrm{d}u$.
Let's check our result evaluating the integral on the right:
$\displaystyle\int_0^1\int_{2u-1}^{2u} \frac{1}{2\sqrt{2u-v}}\mathrm{d}v\mathrm{d}u = \int_0^1 \left[-\sqrt{2u-v}\right]_{v=2u-1}^{v=2u} \mathrm{d}u = \int_0^1 1\cdot\mathrm{d}u = 1.$

b) $$\int_0^1\int_0^1 \mathrm{d}x\mathrm{d}y = \int_{-1}^0\int_0^{\frac{v+1}{2}} \frac{1}{2\sqrt{2u-v}}\mathrm{d}u\mathrm{d}v + \int_0^1\int_{\frac{v}{2}}^{\frac{v+1}{2}} \frac{1}{2\sqrt{2u-v}}\mathrm{d}u\mathrm{d}v + $$
$$+ \int_1^2\int_{\frac{v}{2}}^1 \frac{1}{2\sqrt{2u-v}}\mathrm{d}u\mathrm{d}v.$$

c) When $v = -\dfrac{1}{2}$, $\dfrac{v+1}{2} = \dfrac{1}{4}$ and the inner integral is $\displaystyle\int_0^{1/4} \frac{\mathrm{d}u}{2\sqrt{2u+1}}$.
Since the limits of integration are $0 \le u \le 1/4$, then

i) $P\left(x = u \le \frac{1}{4}\middle| v = -\frac{1}{2}\right) = 1 \qquad$ (always)

ii) $P\left(x = u \ge \frac{1}{2}\middle| v = -1/2\right) = 0 \qquad$ (never)

iii) When $v = \dfrac{1}{2}$, $\dfrac{v}{2} = \dfrac{1}{4}$ and $\dfrac{v+1}{2} = \dfrac{3}{4}$, so the inner integral is

$$\int_{1/4}^{3/4} \frac{1}{2}\left(2u - \frac{1}{2}\right)^{-1/2} \mathrm{d}u = \left[\frac{1}{2}\left(2u - \frac{1}{2}\right)^{1/2}\right]_{1/4}^{3/4} = \frac{1}{2}.$$

Therefore, $P\left(x = u \le \frac{1}{4}\middle| v = \frac{1}{2}\right) = P\left(u < \frac{1}{4}\middle| v = \frac{1}{2}\right) = 0 \qquad$ (never)
(the end point $u = \dfrac{1}{4}$ has probability 0).
Finally, the last probability requires calculation of an integral rather than just knowledge of the limits:

$$P\left(u \geq \frac{1}{2}\middle| v = \frac{1}{2}\right) = \frac{\text{(part)}}{\text{(whole)}} = 1 - \frac{1}{\sqrt{2}} \quad \text{, since}$$

$$\text{(part)} \;=\; \int_{1/2}^{3/4} \frac{1}{2}\left(2u - \frac{1}{2}\right)^{-1/2} \mathrm{d}u \;=\; \left[\frac{1}{2}\left(2u - \frac{1}{2}\right)^{1/2}\right]_{1/2}^{3/4} \;=\; \frac{1}{2} - \frac{1}{2\sqrt{2}}$$

and (whole) $= \frac{1}{2}$ was computed above.

Problem 5

The volume is $\frac{16}{3}$.

In fact the two cylinders are described (in Cartesian coordinates) by
$C_1 : y^2 + z^2 \leq 1$ and $C_2 : x^2 + z^2 \leq 1$.
It is immediate to see that x and y range both between -1 and 1: the problem is to bound z. As the solid is symmetric with respect to four reflections:

- $(x, y, z) \mapsto (-x, y, z)$
- $(x, y, z) \mapsto (x, -y, z)$
- $(x, y, z) \mapsto (x, y, -z)$
- $(x, y, z) \mapsto (y, x, z)$

we can integrate only over $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq x \\ 0 \leq z \leq \sqrt{1 - x^2} \end{cases}$ and then multiply the result by $2^4 = 16$.
The limitation on z is $z \leq \sqrt{1 - x^2}$ because when $y \leq x$, $\sqrt{1 - x^2} \leq \sqrt{1 - y^2}$, that is, in the section $y \leq x$ of the first octant, the surface of C_2 is below the surface of C_1.
Hence $V = 16 \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} \mathrm{d}z\mathrm{d}y\mathrm{d}x = 16 \int_0^1 \int_0^x \sqrt{1 - x^2}\mathrm{d}y\mathrm{d}x =$
$= 16 \int_0^1 x\sqrt{1 - x^2}\mathrm{d}x = 16 \left[\frac{(1 - x^2)^{3/2}}{-3}\right]_0^1 = \frac{16}{3}.$

Problem 6

The average distance is $\overline{d} = \frac{6}{5}a$.

The volume of the sphere is $V = \frac{4}{3}\pi a^3$.
The average distance is $\overline{d} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\varphi} \rho\cdot\rho^2\sin\varphi \mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta =$

$$= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi/2}\frac{(2a\cos\varphi)^4}{4}\sin\varphi\mathrm{d}\varphi\mathrm{d}\theta = \frac{3}{4\pi a^3}\frac{16a^4}{4}\int_0^{2\pi}\left[-\frac{\cos^5\varphi}{5}\right]_0^{\pi/2}\mathrm{d}\theta =$$

$$= \frac{3a}{5\pi}\int_0^{2\pi}\mathrm{d}\theta = \frac{3a}{5\pi}2\pi = \frac{6}{5}a.$$

Problem 7

a) The center of mass $(\overline{x}, \overline{y}, \overline{z})$ has $\overline{x} = \overline{y} = 0$ and

$$\overline{z} = \frac{3}{\pi a^2 h}\int_0^{2\pi}\int_0^{\tan^{-1}(a/h)}\int_0^{h/\cos\varphi}\rho^3\sin\varphi\cos\varphi \ \mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta.$$

Invariance of the cone and δ under $(x, y, z) \mapsto (-x, y, z)$ tells us that $\overline{x} = 0$.
Similarly, the symmetry $(x, y, z) \mapsto (x, -y, z)$ tells us that $\overline{y} = 0$.
The mass is $M = \delta V$.
The integrand is found using $z = \rho\cos\phi$, $M = \frac{\pi a^2 h\delta}{3}$ and
$\mathrm{d}V = \rho^2\sin\varphi \ \mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$.
The limits of integration are found using
flat top: $z = h \iff \rho\cos\phi = h \iff \rho = \frac{h}{\cos\phi}$
cone side: $\phi = \tan(a/h)$.

b) The moment on inertia with respect to the z-axis is

$$I_z = \iiint r^2\delta\mathrm{d}V = \int_0^{2\pi}\int_{\sin^{-1}(b/a)}^{\pi-\sin^{-1}(b/a)}\int_{b/\sin\varphi}^{a}\rho^4\delta\sin^3\varphi \ \mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta.$$

The integrand is found using:
$\mathrm{d}V = \rho^2\sin\varphi \ \mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\theta$ and $r^2 = (\rho\sin\varphi)^2$.
For the limits of integration:
angle: $\sin^{-1}(b/a) < \varphi < \pi - \sin^{-1}(b/a)$ and
lower limit on ρ: $r = b \iff \rho\sin\varphi = b \iff \rho = b/\sin\varphi$.