18.02 Problem Set 8 - Solutions of Part B

Problem 1

a) $\displaystyle\int_{C_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{C_1} y^2 dx + x^2 y dy = \int_0^{2\pi} \sin^2\theta(-2\sin\theta d\theta) + (2\cos\theta)^2 \sin\theta(\cos\theta d\theta)$
$\displaystyle = \int_0^{2\pi} (-2\sin^3\theta + 4\cos^3\theta\sin\theta) d\theta = \left[2\cos\theta - \frac{2}{3}\cos^3\theta - \cos^4\theta\right]_0^{2\pi} = 0.$

The ellipse C_1 is parametrized by $x(\theta) = 2\cos\theta$, $y(\theta) = \sin\theta$ for $0 \le \theta \le 2\pi$. So $dx = -2\sin\theta d\theta$ and $dy = \cos\theta d\theta$.

b) $\displaystyle\int_{C_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \frac{a^4}{12} - \frac{a^3}{3}.$

The integrand $y^2 dx + x^2 y dy$ is equal to zero along the axes.
The segment from $(a, 0)$ to $(0, a)$ is parametrized as:
$x = a - y$ for $0 \le y \le a$, so that $dx = -dy$.
The line integral along this segment is $\displaystyle\int_0^a y^2 dx + x^2 y dy =$
$\displaystyle = \int_0^a y^2(-dy) + (a-y)^2 y dy = \int_0^a (y^3 - (2a+1)y^2 + a^2 y) dy =$
$\displaystyle = \left[\frac{1}{4}y^4 - \frac{2a+1}{3}y^3 + \frac{a^2}{2}y^2\right]_0^a = \frac{a^4}{4} - \frac{(2a+1)a^3}{3} + \frac{a^4}{2}.$

Problem 2

a) $\displaystyle\nabla\theta = \left\langle \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y} \right\rangle = \left\langle \frac{-y/x^2}{1+(y/x)^2}, \frac{1/x}{1+(y/x)^2} \right\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$

b) $\displaystyle\int_C \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, ds = \theta(B) - \theta(A) = \theta_2 - \theta_1.$
Consider the curve C defined by: $x(t) = 1$, $y(t) = -1 + t$ for $0 \le t \le 2$.
For this curve $B = (1, -1)$ and $A = (1, 1)$, so $\theta_2 = -\pi/4$ and $\theta_1 = \pi/4$.
Hence, the line integral along this C is $-\pi/2$.

c) $\int_{C_1} \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s = \pi$ and $\int_{C_2} \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s = -\pi$.

Along C_1: $\hat{\mathbf{T}} = -y\hat{\imath} + x\hat{\jmath}$ and $\vec{\mathbf{F}} = -y\hat{\imath} + x\hat{\jmath}$, so $\vec{\mathbf{F}} \cdot \hat{\mathbf{T}} = 1$.
Hence, $\int_{C_1} \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s = \int_{C_1} \mathrm{d}s = \text{length}(C_1) = \pi$.
Along C_2: $\hat{\mathbf{T}} = y\hat{\imath} - x\hat{\jmath}$ and $\vec{\mathbf{F}} = -y\hat{\imath} + x\hat{\jmath}$, so $\vec{\mathbf{F}} \cdot \hat{\mathbf{T}} = -1$.
Hence, $\int_{C_2} \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s = -\int_{C_2} \mathrm{d}s = -\text{length}(C_2) = -\pi$.

d) $\vec{\mathbf{F}}$ is conservative in R_2 and R_3 but not in R_1.

It is conservative in R_3 because we found a potential in (a).
It is also conservative in R_2: we can define a polar angle function θ on R_2 in such a way that $-\pi < \theta(x, y) < \pi$.
In R_1 it is not conservative, because we found in (c) two paths C_1 and C_2 between $(1, 0)$ and $(0, 1)$ such that $\int_{C_1} \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s \neq \int_{C_2} \vec{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s$.

Explanation: The fundamental theorem of calculus for line integrals implies that, if $\vec{\mathbf{F}}$ is conservative on R_2 (so that there exists a well-defined and differentiable potential f in R_2 such that $\vec{\mathbf{F}} = \nabla f$), then for every simple closed curve C *totally contained inside* R_2 we have $\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.
As $\vec{\mathbf{F}}$ is not conservative in R_1, then we can find a simple closed curve C' *contained in* R_1 such that $\int_{C'} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \neq 0$.
Thus, the fact that $\vec{\mathbf{F}}$ is conservative in R_2 but not in R_1 just implies that the curve C' cannot be totally contained in R_2.
(C' cannot be totally contained inside R_3 either, because $\vec{\mathbf{F}}$ is conservative on R_3.)

Problem 3

a) $\text{curl}\vec{\mathbf{F}} = (yr^n)_x - (xr^n)_y = nyr^{n-1}r_x - nxr^{n-1}r_y = nyxr^{n-2} - nxyr^{n-2} = 0$ for every n.

We used that $r = \sqrt{x^2+y^2}$, so $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial r}{\partial y} = \frac{y}{r}$.

b) $g(r) = \begin{cases} \frac{r^{n+2}}{n+2} & \text{if } n \neq -2 \\ \ln r & \text{if } n = -2 \end{cases}$.

We want $\frac{\partial}{\partial x} g(r) = xr^n$ and $\frac{\partial}{\partial y} g(r) = yr^n$.
$\frac{\partial}{\partial x} g(r) = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}$ and $\frac{\partial}{\partial y} g(r) = g'(r)\frac{y}{r}$.
Hence $g'(r) = r^{n+1}$.

Problem 4

a) If $\overrightarrow{\mathbf{F}} = y^2\boldsymbol{\hat\imath} + x^2y\boldsymbol{\hat\jmath}$ is a gradient, then $\frac{\partial}{\partial x}(x^2y) = \frac{\partial}{\partial y}(y^2)$,
but $(x^2y)_x = 2xy \neq 2y = (y^2)_y$.

b) We try to find a potential g setting $g(x_0, y_0) = \int_C \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s$, where C is the path that goes $(0,0)$ to $(x_0, 0)$ along the x-axis and from $(x_0, 0)$ to (x_0, y_0) parallel to the y-axis.
Then $\int_C \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{T}}\, \mathrm{d}s = \int_0^{y_0} x_0^2 y\, \mathrm{d}y = \frac{1}{2}x_0^2 y_0^2$.
However, $\nabla g(x,y) = \langle xy^2, x^2y \rangle \neq \overrightarrow{\mathbf{F}}$.
If we try to use a different path, the result changes but we never get $\nabla g = \overrightarrow{\mathbf{F}}$.

c) As we want $\nabla g = \overrightarrow{\mathbf{F}}$, we start with $g_x(x,y) = y^2$.
It gives $g(x,y) = xy^2 + h(y)$.
Then $g_y(x,y) = x^2y$ gives $2xy + h'(y) = x^2y$.
There does not exist any $h(y)$ (which does not depends on x) such that
$h'(y) = x^2y - 2xy$.