18.02 Problem Set 2 - Solutions of Part B

Problem 1

a) The position vector is $\overrightarrow{\mathbf{r}}(t) = \langle \cos(\pi - t), \sin(\pi - t) \rangle = \langle -\cos(t), \sin(t) \rangle$.

In fact, the position vector of a uniform circular motion centered at the origin is given by

$$\overrightarrow{\mathbf{r}}(t) = \langle R\cos(at + b), R\sin(at + b) \rangle$$

where $R > 0$ is the radius.
Its velocity vector is

$$\overrightarrow{\mathbf{v}}(t) = \langle -aR\sin(at + b), aR\cos(at + b) \rangle$$

and so the speed is $|\overrightarrow{\mathbf{v}}(t)| = R|a|$.
Now $\overrightarrow{\mathbf{r}}(0) = \langle R\cos(b), R\sin(b) \rangle = \langle -1, 0 \rangle$.
This forces $R = 1$ and $b = \pi$ (or $(2k + 1)\pi$ with k integer).
The condition on the speed $|\overrightarrow{\mathbf{v}}(t)| = 1$ implies $a = \pm 1$.
As the motion is clockwise, then $a = -1$.
Concluding we get $\overrightarrow{\mathbf{r}}(t) = \langle \cos(\pi - t), \sin(\pi - t) \rangle$.

b) The position vector is $\overrightarrow{\mathbf{r}}(t) = \langle 10\cos(6t), 10\sin(6t) \rangle$.

Similarly to (a), $\overrightarrow{\mathbf{r}}(t) = \langle R\cos(b), R\sin(b) \rangle = \langle 10, 0 \rangle$, so that $R = 10$ and $b = 0$ (or $2k\pi$ with k integer).
Then $|\overrightarrow{\mathbf{v}}(t)| = 60$ implies $10|a| = 60$, so that $a = 6$ because the motion is counterclockwise.

c) The position vector is $\overrightarrow{\mathbf{r}}(t) = \langle 10\cos(120\pi t), 10\sin(120\pi t) \rangle$.

As in (b), we have $R = 10$ and $b = 0$.
Moreover, 60 rotations per minute means an angle of $60 \cdot 2\pi = 120\pi$ radiants per minute.

d) The position vector is $\overrightarrow{\mathbf{r}}(t) = \langle 1 - \cos(t) - t, 1 + \sin(t) - t, \frac{1}{2}t^2 \rangle$.

In fact, $\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{\mathbf{v}}(t) = \langle \cos(t), -\sin(t), 1 \rangle$ implies that

$$\overrightarrow{\mathbf{v}}(t) = \langle a + \sin(t), b + \cos(t), c + t \rangle.$$

But $\overrightarrow{\mathbf{v}}(0) = \langle -1, 0, 0\rangle$, so that $a = -1$, $b = 1$ and $c = 0$.
Hence $\overrightarrow{\mathbf{v}}(t) = \langle \sin(t) - 1, \cos(t) - 1, t\rangle$.
Similarly, $\dfrac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{\mathbf{r}}(t) = \langle \sin(t) - 1, \cos(t) - 1, t\rangle$ implies that

$$\overrightarrow{\mathbf{r}}(t) = \langle a' - \cos(t) - t, b' + \sin(t) - t, c' + \frac{1}{2}t^2\rangle.$$

But $\overrightarrow{\mathbf{r}}(0) = \langle 0, 1, 0\rangle$ forces $a' = 1$, $b' = 1$ and $c' = 0$.
Hence $\overrightarrow{\mathbf{r}}(t) = \langle 1 - \cos(t) - t, 1 + \sin(t) - t, \frac{1}{2}t^2\rangle$.

Problem 2

a) The hypothesis is that $|\overrightarrow{\mathbf{r}}(t)| = R$ for all values of t, where R is some positive number (the radius of the sphere).
Taking the square of both sides we get

$$\overrightarrow{\mathbf{r}}(t) \cdot \overrightarrow{\mathbf{r}}(t) = |\overrightarrow{\mathbf{r}}(t)|^2 = R^2.$$

Differentiating both sides with respect to t we get

$$\frac{\mathrm{d}\overrightarrow{\mathbf{r}}(t)}{\mathrm{d}t} \cdot \overrightarrow{\mathbf{r}}(t) + \overrightarrow{\mathbf{r}}(t) \cdot \frac{\mathrm{d}\overrightarrow{\mathbf{r}}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\overrightarrow{\mathbf{r}}(t) \cdot \overrightarrow{\mathbf{r}}(t)\right) = 0$$

which means $\overrightarrow{\mathbf{v}}(t) \cdot \overrightarrow{\mathbf{r}}(t) = 0$ for all t.
We proved that $\overrightarrow{\mathbf{v}}(t)$ is orthogonal to $\overrightarrow{\mathbf{r}}(t)$ at all times, thus $\overrightarrow{\mathbf{v}}(t)$ is tangent to the sphere of radius R centered at the origin.

b) We have to check that $|\overrightarrow{\mathbf{r}}(t)| = 1$ for all t. We have

$$|\overrightarrow{\mathbf{r}}(t)|^2 = \cos^2(t)\sin^2(2t) + \sin^2(t)\sin^2(2t) + \cos^2(2t) = \sin^2(2t) + \cos^2(2t) = 1$$

so that $|\overrightarrow{\mathbf{r}}(t)| = 1$.

c) $\overrightarrow{\mathbf{v}}(t) = \langle -\sin(t)\sin(2t) + 2\cos(t)\cos(2t), \cos(t)\sin(2t) + 2\sin(t)\cos(2t), -2\sin(2t)\rangle$
just differentiating $\overrightarrow{\mathbf{r}}(t)$ componentwise.

d) The angle is $\theta = \arccos(1/\sqrt{5})$ (or $\arccos(-1/\sqrt{5}) = \pi - \arccos(1/\sqrt{5})$).
The points of intersection between the trajectory and the equator are
$(\sqrt{2}/2, \sqrt{2}/2, 0), (\sqrt{2}/2, -\sqrt{2}/2, 0), (-\sqrt{2}/2, -\sqrt{2}/2, 0), (-\sqrt{2}/2, \sqrt{2}/2, 0)$.

First we have to find the point(s) where the trajectory intersects the equator, that is where

$$z(t) = \cos(2t) = 0.$$

It happens for $t = \pi/4 + k\pi/2$ with k integer. These values of t determine four points on the sphere:

$$P_1 = (\sqrt{2}/2, \sqrt{2}/2, 0) \text{ (corresponding to } t = \pi/4)$$

$$P_2 = (\sqrt{2}/2, -\sqrt{2}/2, 0) \text{ (corresponding to } t = 3\pi/4)$$

$$P_3 = (-\sqrt{2}/2, -\sqrt{2}/2, 0) \text{ (corresponding to } t = 5\pi/4)$$

$$P_4 = (-\sqrt{2}/2, \sqrt{2}/2, 0) \text{ (corresponding to } t = 7\pi/4).$$

Notice that, if P is a point on the unit circle in the plane, then a tangent vector at P to the circle can be obtained rotating $\overrightarrow{OP}$ by $\pi/2$ (or by $-\pi/2$, in this case we would get the opposite vector, which is still tangent to the equator).
As a consequence, a vector $\overrightarrow{w}_1$ tangent at P_1 to the equator is

$$\overrightarrow{w}_1 = \langle -\sqrt{2}/2, \sqrt{2}/2, 0 \rangle$$

and a vector $\overrightarrow{w}_2$ tangent at P_2 to the equator is

$$\overrightarrow{w}_2 = \langle \sqrt{2}/2, \sqrt{2}/2, 0 \rangle.$$

Similarly we get

$$\overrightarrow{w}_3 = \langle \sqrt{2}/2, -\sqrt{2}/2, 0 \rangle \qquad \text{and} \qquad \overrightarrow{w}_4 = \langle -\sqrt{2}/2, -\sqrt{2}/2, 0 \rangle.$$

Using the formula for the velocity of the given trajectory from (c), we can compute its velocity vectors $\overrightarrow{v}_i$ at P_i for $i = 1, 2, 3, 4$. We get

$$\overrightarrow{v}_1 = \overrightarrow{\mathbf{v}}(\pi/4) = \langle -\sqrt{2}/2, \sqrt{2}/2, -2 \rangle \qquad \overrightarrow{v}_2 = \overrightarrow{\mathbf{v}}(3\pi/4) = \langle \sqrt{2}/2, \sqrt{2}/2, 2 \rangle.$$

$$\overrightarrow{v}_3 = \overrightarrow{\mathbf{v}}(5\pi/4) = \langle \sqrt{2}/2, -\sqrt{2}/2, -2 \rangle \qquad \overrightarrow{v}_4 = \overrightarrow{\mathbf{v}}(7\pi/4) = \langle -\sqrt{2}/2, -\sqrt{2}/2, 2 \rangle.$$

Call θ_i one of the two angles between the trajectory and the equator at P_i for $i = 1, 2, 3, 4$. Then

$$\cos(\theta_1) = \frac{\overrightarrow{v}_1 \cdot \overrightarrow{w}_1}{|\overrightarrow{v}_1|\,|\overrightarrow{w}_1|} = \frac{1}{\sqrt{5}} \qquad \cos(\theta_2) = \frac{\overrightarrow{v}_2 \cdot \overrightarrow{w}_2}{|\overrightarrow{v}_2|\,|\overrightarrow{w}_2|} = \frac{1}{\sqrt{5}}$$

$$\cos(\theta_3) = \frac{\overrightarrow{v}_3 \cdot \overrightarrow{w}_3}{|\overrightarrow{v}_3|\,|\overrightarrow{w}_3|} = \frac{1}{\sqrt{5}} \qquad \cos(\theta_4) = \frac{\overrightarrow{v}_4 \cdot \overrightarrow{w}_4}{|\overrightarrow{v}_4|\,|\overrightarrow{w}_4|} = \frac{1}{\sqrt{5}}.$$