18.02 Problem Set 6 - Solutions of Part B

Problem 1

a) The parametric equations are $\begin{cases} x = x_0 + \dfrac{a}{c}\Delta z \\ y = y_0 + \dfrac{b}{c}\Delta z \\ z = z_0 + \Delta z \end{cases}$

The condition is $c \neq 0$.

In fact, the standard parametric equations (with parameter t) would be $P(t) = P_0 + t\overrightarrow{\mathbf{v}}$, that is in components $\begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases}$.

From the third equation, if $c \neq 0$, we get $t = \dfrac{\Delta z}{c}$.

b) $\dfrac{\Delta \overrightarrow{\mathbf{r}}}{\Delta z} = \langle \dfrac{a}{c}, \dfrac{b}{c}, 1 \rangle$ and $\dfrac{\mathrm{d}\overrightarrow{\mathbf{r}}}{\mathrm{d}z}(P_0) = \langle \dfrac{a}{c}, \dfrac{b}{c}, 1 \rangle$.

In fact, $\dfrac{\Delta \overrightarrow{\mathbf{r}}}{\Delta z} = \langle \dfrac{\Delta x}{\Delta z}, \dfrac{\Delta y}{\Delta z}, \dfrac{\Delta z}{\Delta z} \rangle = \langle \dfrac{a}{c}, \dfrac{b}{c}, 1 \rangle$.

Being constant, $\dfrac{\Delta \overrightarrow{\mathbf{r}}}{\Delta z}$ coincides with the derivative $\dfrac{\mathrm{d}\overrightarrow{\mathbf{r}}}{\mathrm{d}z}$ along the line at any point of the line.

c) Parametric equations $\begin{cases} x(t) = 1 - 4t \\ y(t) = 1 - 5t \\ z(t) = 1 - t \end{cases}$

i) $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial z}\right)_{u,v} = \langle 4, 5, 1 \rangle$

ii) $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial y}\right)_{u,v} = \langle \dfrac{4}{5}, 1, \dfrac{1}{5} \rangle$

iii) $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial x}\right)_{u,v} = \langle 1, \dfrac{5}{4}, \dfrac{1}{4} \rangle$

In fact, $\langle 1, -1, 1 \rangle$ is a normal vector to any plane given by $u = x - y + z = c_1$ and $\langle -2, 1, 3 \rangle$ is a normal vector to any plane given by $v = -2x + y + 3z = c_2$. Hence, their intersection is a line parallel to the vector

$\langle 1,-1,1\rangle \times \langle -2,1,3\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ -2 & 1 & 3 \end{vmatrix} = -4\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and the parametric equation is $P(t) = (1,1,1) + t\langle -4,-5,-1\rangle$.

Following (b), we obtain $\dfrac{\Delta \overrightarrow{\mathbf{r}}}{\Delta z} = \langle 4,5,1\rangle$, $\dfrac{\Delta \overrightarrow{\mathbf{r}}}{\Delta y} = \langle \frac{4}{5}, 1, \frac{1}{5}\rangle$ and $\dfrac{\Delta \overrightarrow{\mathbf{r}}}{\Delta x} = \langle 1, \frac{5}{4}, \frac{1}{4}\rangle$, which hold along the line $\begin{cases} u = 1 \\ v = 2 \end{cases}$ (and, in fact, along any line obtained intersecting a plane $u = c_1$ and a plane $v = c_2$).

This means that $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial z}\right)_{u,v} = \langle 4,5,1\rangle$, $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial y}\right)_{u,v} = \langle \frac{4}{5}, 1, \frac{1}{5}\rangle$ and $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial x}\right)_{u,v} = \langle 1, \frac{5}{4}, \frac{1}{4}\rangle$.

d) The lines are all parallel (because they are parallel to $\langle -4,-5,-1\rangle$).

Given a curve C a point P on it, the derivative $\dfrac{\mathrm{d}x}{\mathrm{d}z}(P)$ is the rate of change of x with respect to z along C (in other words, it is also the slope of the curve obtained by projecting C onto the xz-plane).

In our case, the derivative $\dfrac{\mathrm{d}x}{\mathrm{d}z}$ along a line $\begin{cases} u = c_1 \\ v = c_2 \end{cases}$ coincides with $\left(\dfrac{\partial x}{\partial z}\right)_{u,v}$, so that the derivative $\dfrac{\mathrm{d}x}{\mathrm{d}z} = \left(\dfrac{\partial x}{\partial z}\right)_{u,v}$ is constant along the line.

Moreover, if we vary c_1 and c_2, we obtain parallel lines, so $\Delta z/\Delta x$ will be the same at any point (in other words, their projections will have the same slope).

An analogous argument shows that the other derivatives $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial x}\right)_{u,v}$, $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial y}\right)_{u,v}$ and $\left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial z}\right)_{u,v}$ are constant.

e) $\left(\dfrac{\partial w}{\partial x}\right)_{u,v} = \nabla w \cdot \left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial x}\right)_{u,v} = f_x + \dfrac{5}{4} f_y + \dfrac{1}{4} f_z$

$\left(\dfrac{\partial w}{\partial y}\right)_{u,v} = \nabla w \cdot \left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial y}\right)_{u,v} = \dfrac{4}{5} f_x + f_y + \dfrac{1}{5} f_z$

$\left(\dfrac{\partial w}{\partial z}\right)_{u,v} = \nabla w \cdot \left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial z}\right)_{u,v} = 4 f_x + 5 f_y + f_z$

In fact, using the chain rule $\left(\dfrac{\partial w}{\partial x}\right)_{u,v} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\left(\dfrac{\partial y}{\partial x}\right)_{u,v} + \dfrac{\partial f}{\partial z}\left(\dfrac{\partial z}{\partial x}\right) =$
$= \nabla w \cdot \left(\dfrac{\partial \overrightarrow{\mathbf{r}}}{\partial x}\right)_{u,v} = \nabla w \cdot \langle 1, \frac{5}{4}, \frac{1}{4}\rangle = f_x + \dfrac{5}{4} f_y + \dfrac{1}{4} f_z$.

Similarly for the other derivatives.

f) A tangent vector to the curve at $(1,2,4)$ is $\overrightarrow{\mathbf{v}} = \langle -208, 218, 84\rangle$.
Along the curve, $\dfrac{\mathrm{d}w}{\mathrm{d}z}(1,2,4) = -\dfrac{52}{21}f_x + \dfrac{109}{42}f_y + f_z$ and
$\dfrac{\mathrm{d}w}{\mathrm{d}x}(1,2,4) = f_x - \dfrac{109}{104}f_y - \dfrac{21}{52}f_z$.
If P is any point on the curve, then the derivative $\dfrac{\mathrm{d}w}{\mathrm{d}x}(P)$ along the curve coincides with $\left(\dfrac{\partial w}{\partial x}\right)_{w_1,w_2}(P)$ and $\dfrac{\mathrm{d}w}{\mathrm{d}z}(P)$ along the curve coincides with $\left(\dfrac{\partial w}{\partial z}\right)_{w_1,w_2}(P)$.

In fact, $\nabla w_1 = \langle 3x^2 - yz, -xz, -xy\rangle$ and $\nabla w_2 = \langle 1, -2yz, -y^2+3z^2\rangle$.
At $(1,2,4)$ we find $\nabla w_1(1,2,4) = \langle -5,-4,-2\rangle$ and $\nabla w_2(1,2,4) = \langle 1,-16,44\rangle$.
Hence a vector parallel to the curve $\begin{cases} w_1 = -7 \\ w_2 = 49 \end{cases}$ at $(1,2,4)$ is

$$\langle -5,-4,-2\rangle \times \langle 1,-16,44\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5 & -4 & -2 \\ 1 & -16 & 44 \end{vmatrix} = -208\hat{\mathbf{i}} + 218\hat{\mathbf{j}} + 84\hat{\mathbf{k}}.$$

Therefore, along the curve we have $\dfrac{\mathrm{d}w}{\mathrm{d}z}(1,2,4) = f_x(1,2,4)\dfrac{-208}{84} + f_y(1,2,4)\dfrac{218}{84} + f_z(1,2,4)$ and $\dfrac{\mathrm{d}w}{\mathrm{d}x}(1,2,4) = f_x(1,2,4) + f_y(1,2,4)\dfrac{218}{-208} + \dfrac{84}{-208}f_z(1,2,4)$.

Problem 2

a) Let $(0,0), (x_0,0), (x_0,y_0), (0,y_0)$ be the vertices of the rectangular base. The volume of the prism is

$$V = \int_0^{y_0}\int_0^{x_0} z\,\mathrm{d}x\mathrm{d}y = \int_0^{y_0}\int_0^{x_0}(ax+by+c)\mathrm{d}x\mathrm{d}y = \int_0^{y_0}\left[\frac{a}{2}x^2 + bxy + cx\right]_0^{x_0}\mathrm{d}y = \int_0^{y_0}\frac{a}{2}x_0^2 + bx_0y + cx_0\mathrm{d}y = \left[\frac{a}{2}x_0^2y + \frac{b}{2}x_0y^2 + cx_0y\right]_0^{y_0} = x_0y_0\frac{ax_0+by_0+2c}{2}.$$

The lengths of the four vertical edges are: $z(0,0) = c$, $z(x_0,0) = ax_0+c$, $z(x_0,y_0) = ax_0+by_0+c$ and $z(0,y_0) = by_0+c$.
Hence the average of the lengths of the four vertical edges is

$$\ell = \frac{c + (ax_0+c) + (ax_0+by_0+c) + (by_0+c)}{4} = \frac{2c+ax_0+by_0}{2}.$$

The area of the base is clearly $A = x_0y_0$.
Therefore $V = A\cdot\ell$.

Problem 3

a) $\displaystyle\int_1^a e^{-xy}\mathrm{d}y = \left[\frac{e^{-xy}}{-x}\right]_{y=1}^{y=a} = \frac{e^{-x}-e^{-ax}}{x}$

b) $\displaystyle\int_0^{+\infty} \frac{e^{-x}-e^{-ax}}{x}\mathrm{d}x = \int_0^{+\infty}\int_1^a e^{-xy}\mathrm{d}y\mathrm{d}x = \int_1^a\int_0^{+\infty} e^{-xy}\mathrm{d}x\mathrm{d}y =$
$\displaystyle= \int_1^a \left[\frac{e^{-xy}}{-y}\right]_{x=0}^{x=+\infty}\mathrm{d}y = \int_1^a \frac{1}{y}\mathrm{d}y = [\ln(y)]_1^a = \ln(a)$

Problem 4

a) The mass is $m = \dfrac{\pi}{8}$.
The centroid is $(\overline{x},\overline{y}) = (0, \dfrac{32}{15\pi})$.
The moment of inertia with respect to the x-axis is $I_x = \dfrac{\pi}{16}$.
The moment of inertia with respect to the y-axis is $I_y = \dfrac{\pi}{48}$.
The polar moment of inertia is $I_0 = \dfrac{\pi}{12}$.

In fact, the domain D is described in polar coordinates by $\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \pi \end{cases}$ and the density in polar coordinates is $\delta(r,\theta) = r^2\sin^2\theta$.
The mass is given by $\displaystyle m = \iint_D \delta\mathrm{d}A = \int_0^\pi\int_0^1 r^2\sin^2\theta\cdot r\mathrm{d}r\mathrm{d}\theta =$
$\displaystyle= \int_0^\pi \sin^2\theta\left[\frac{r^4}{4}\right]_0^1\mathrm{d}\theta = \frac{1}{4}\int_0^\pi \frac{1-\cos(2\theta)}{2}\mathrm{d}\theta = \frac{1}{4}\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^\pi = \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{8}$

Because D and δ are symmetric with respect to the y-axis (that is, they are invariant under the transformation $x \mapsto -x$), then the x-coordinate of the centroid $\overline{x} = 0$.
Instead $\displaystyle\overline{y} = \frac{\iint_D y\delta\mathrm{d}A}{m} = \frac{8}{\pi}\int_0^\pi\int_0^1 r^3\sin^3\theta\cdot r\mathrm{d}r\mathrm{d}\theta = \frac{8}{\pi}\int_0^\pi \sin^3\theta\left[\frac{r^5}{5}\right]_0^1\mathrm{d}\theta =$
$\displaystyle= \frac{8}{5\pi}\int_0^\pi \sin\theta(1-\cos^2\theta)\mathrm{d}\theta = \frac{8}{5\pi}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^\pi = \frac{8}{5\pi}\frac{4}{3} = \frac{32}{15\pi}$

$\displaystyle I_x = \iint_D y^2\delta\mathrm{d}A = \int_0^\pi\int_0^1 r^4\sin^4\theta\cdot r\mathrm{d}r\mathrm{d}\theta = \int_0^\pi \sin^4\theta\left[\frac{r^6}{6}\right]_0^1\mathrm{d}\theta = \frac{1}{6}\int_0^\pi \sin^4\theta\mathrm{d}\theta$

Now, notice that $\sin^4\theta = (\sin^2\theta)^2 = \dfrac{(1-\cos 2\theta)^2}{4} = \dfrac{1}{4}(1-2\cos 2\theta+\cos^2 2\theta) =$
$= \dfrac{1}{4}\left(1-2\cos 2\theta + \dfrac{1+\cos 4\theta}{2}\right) = \dfrac{1}{8}(3-4\cos 2\theta+\cos 4\theta)$
Hence $I_x = \dfrac{1}{48}\displaystyle\int_0^\pi (3-4\cos 2\theta+\cos 4\theta)\mathrm{d}\theta = \dfrac{1}{48}\left[3\theta - 2\sin 2\theta + \dfrac{\sin 4\theta}{4}\right]_0^\pi = \dfrac{\pi}{16}$

[In the calculations above, we could have used 3B from the Notes, which tells us that $\displaystyle\int_0^\pi \sin^3\theta\mathrm{d}\theta = 2\int_0^{\pi/2}\sin^3\theta\mathrm{d}\theta = \dfrac{4}{3}$ and $\displaystyle\int_0^\pi \sin^4\theta\mathrm{d}\theta = 2\int_0^{\pi/2}\sin^4\theta\mathrm{d}\theta = \dfrac{3\pi}{8}$.]

$I_y = \displaystyle\iint_D x^2\delta\mathrm{d}A = \int_0^\pi\int_0^1 r^4\sin^2\theta\cos^2\theta\cdot r\mathrm{d}r\mathrm{d}\theta = \dfrac{1}{6}\int_0^\pi \dfrac{1}{4}(2\sin\theta\cos\theta)^2\mathrm{d}\theta =$
$= \dfrac{1}{24}\displaystyle\int_0^\pi \sin^2 2\theta\mathrm{d}\theta = \dfrac{1}{24}\int_0^\pi \dfrac{1-\cos 4\theta}{2}\mathrm{d}\theta = \dfrac{1}{48}\left[\theta - \dfrac{\sin 4\theta}{4}\right]_0^\pi = \dfrac{\pi}{48}$

The polar moment of inertia is obtained from I_x and I_y in the following way: $I_0 = \displaystyle\iint_D (x^2+y^2)\delta\mathrm{d}A = I_x + I_y = \dfrac{\pi}{16} + \dfrac{\pi}{48} = \dfrac{3\pi+\pi}{48} = \dfrac{\pi}{12}$.