18.02 Problem Set 5 - Solutions of Part B

Problem 1

a) The system of equations is $\begin{cases} z - 2x^2 - y^2 &= 0 \\ 2(x-2) &= -4x\lambda \\ 2(y-1) &= -2y\lambda \\ 2(z-10) &= \lambda \end{cases}$

Let $P_0 = (2, 1, 10)$ and $g(x, y, z) = z - 2x^2 - y^2$.
Call $P = (x, y, z)$ a generic point. We want to minimize the distance between P and P_0, or equivalently $f(x, y, z) = |\overrightarrow{P_0P}|^2 = (x-2)^2 + (y-1)^2 + (z-10)^2$, with the constraint $g(x, y, z) = 0$.
Introducing the Lagrange multiplier λ, we obtain the following system of equations

$$\begin{cases} g(x,y,z) &= 0 \\ \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) \end{cases}$$

Computing the gradients, we have $\nabla f = \langle 2(x-2), 2(y-1), 2(z-10)\rangle$ and $\nabla g = \langle -4x, -2y, 1\rangle$.

b) The point $P = (x, y, z)$ has coordinates (approximated to 1/10000): $x = 2.1132$, $y = 1.0275$ and $z = 9.9866$.
The coordinated of the point P computed in *Problem 6b - PS4* using linear approximation were (approximately) $x = 2.116$, $y = 1.029$, $z = 9.986$, so **within** 1/100 of the exact answer.

Solving the system in (a), we get

$$\begin{cases} z - 2x^2 - y^2 = 0 \\ x = \dfrac{2}{1+2\lambda} \\ y = \dfrac{1}{1+\lambda} \\ z = \dfrac{20+\lambda}{2} \end{cases}$$

Substituting inside the first equation we have

$$(20+\lambda)(1+2\lambda)^2(1+\lambda)^2 - 16(1+\lambda)^2 - 2(1+2\lambda)^2 = 0$$

and finally

$$4\lambda^5 + 92\lambda^4 + 253\lambda^3 + 242\lambda^2 + 81\lambda + 2 = 0$$

This equation has three real solutions.

- $\lambda = 0.02677$, which gives $x = 2.11315$, $y = 1.02751$ and $z = 9.98661$
- $\lambda = -1.38482$, which gives $x = -1.13018$, $y = -2.59865$ and $z = 0.30759$
- $\lambda = -19.98391$, which gives $x = -0.05132$, $y = -0.05268$ and $z = 0.00804$

Clearly the first one corresponds to P closest to P_0.
The solution coming from linear approximation in *Problem 6b - PS4* was $x = 146/69 \approx 2.116$, $y = 71/69 \approx 1.029$ and $z = 689/69 \approx 9.986$.

Problem 2

a) $\left(\frac{\partial w}{\partial x}\right)_z = f_x - \frac{g_x}{g_y} f_y$.
Instead $\left(\frac{\partial w}{\partial x}\right)_x$ and $\left(\frac{\partial w}{\partial x}\right)_y$ do not make sense.
In fact, $\left(\frac{\partial w}{\partial x}\right)_x$ is certainly meaningless because we cannot differentiate with respect to x if x is fixed!
Moreover, the relation $g(x, y) = c$ implies that

$$\frac{\partial g}{\partial x}\mathrm{d}x + \frac{\partial g}{\partial y}\mathrm{d}y = 0$$

As a consequence, fixing x and so setting $\mathrm{d}x = 0$ forces also $\mathrm{d}y$ to be zero (except in some trivial cases when $g(x, y)$ really depends only on x).
Hence $\left(\frac{\partial w}{\partial x}\right)_y$ is meaningless too.
On the contrary, we can compute $\left(\frac{\partial w}{\partial x}\right)_z$ using differentials.
The relation $g(x, y) = c$ gives us

$$\mathrm{d}y = -\frac{g_x}{g_y}\mathrm{d}x$$

Totally differentiating $w = f(x, y, z)$ we have

$$\mathrm{d}w = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

We are keeping z fixed, so $\mathrm{d}z = 0$. Substituting we obtain

$$\mathrm{d}w = f_x\mathrm{d}x - f_y\frac{g_x}{g_y}\mathrm{d}x$$

and so the result.

b) $\left(\frac{\partial w}{\partial t}\right)_x = x\tan(x+y) + xy.$

Totally differentiating $\sin(x+y) = 4t$ we have $\cos(x+y)\mathrm{d}x + \cos(x+y)\mathrm{d}y = 4\mathrm{d}t$ and so $\mathrm{d}y = \frac{4}{\cos(x+y)}\mathrm{d}t$, because $\mathrm{d}x = 0$.
Totally differentiating $w = xyt - 2$ we get $\mathrm{d}w = yt\mathrm{d}x + xt\mathrm{d}y + xy\mathrm{d}t$. Plugging the previous relation in, we obtain

$$\mathrm{d}w = xt\frac{4}{\cos(x+y)}\mathrm{d}t + xy\mathrm{d}t = \left(x\tan(x+y) + xy\right)\mathrm{d}t$$

which gives our result.

c) Along the curve $\frac{\mathrm{d}z}{\mathrm{d}y}(1,2,4) = \frac{42}{109}$.

The curve is given by the system of equations

$$(*) \qquad \begin{cases} x^3 - zyx &= -7 \\ x - y^2z + z^3 &= 49 \end{cases}$$

so the differentials $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ along the curve satisfy the following system of linear equations

$$\begin{cases} (3x^2 - yz)\ \mathrm{d}x - xz\ \mathrm{d}y - xy\ \mathrm{d}z = 0 \\ \mathrm{d}x - 2yz\ \mathrm{d}y + (3z^2 - y^2)\ \mathrm{d}z = 0 \end{cases}$$

obtained by totally differentiating $(*)$.
From the second equation we can extract $\mathrm{d}x$:

$$\mathrm{d}x = (2yz)\mathrm{d}y + (y^2 - 3z^2)\mathrm{d}z$$

Plugging it into the first equation we get

$$(3x^2 - yz)[(2yz)\mathrm{d}y + (y^2 - 3z^2)\mathrm{d}z] - xz\mathrm{d}y - xy\mathrm{d}z = 0$$
$$(6x^2yz - 2y^2z^2 - xz)\mathrm{d}y + (3x^2y^2 - 9x^2z^2 + 3yz^3 - y^3z - xy)\mathrm{d}z = 0$$

Setting $x = 1$, $y = 2$ and $z = 4$ we obtain

$$(6\cdot 2\cdot 4 - 2\cdot 2^2\cdot 4^2 - 4)\mathrm{d}y + (3\cdot 2^2 - 9\cdot 4^2 + 3\cdot 2\cdot 4^3 - 2^3\cdot 4 - 2)\mathrm{d}z = 0$$
$$(48 - 128 - 4)\mathrm{d}y + (12 - 144 + 384 - 32 - 2)\mathrm{d}z = 0$$
$$-84\mathrm{d}y + 218\mathrm{d}z = 0$$

from which we conclude that $\frac{\mathrm{d}z}{\mathrm{d}y}(1,2,4) = \frac{42}{109}$ along the curve.