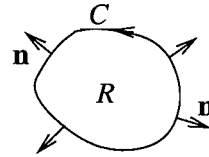# V4. Green's Theorem in Normal Form

1. Green's theorem for flux.

Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ represent a two-dimensional flow field, and C a simple closed curve, positively oriented, with interior R.

According to the previous section,

$$(1) \qquad \text{flux of } F \text{ across } C \;=\; \oint_C M\,dy - N\,dx\;.$$

Notice that since the normal vector points outwards, away from R, the flux is positive where the flow is out of R; flow into R counts as negative flux.

We now apply Green's theorem to the line integral in (1); first we write the integral in standard form (dx first, then dy):

$$\oint_C M\,dy - N\,dx \;=\; \oint_C -N\,dx + M\,dy \;=\; \iint_R \left(M_x - (-N)_y\right) dA\;.$$

This gives us **Green's theorem in the normal form**

$$(2) \qquad \oint_C M\,dy - N\,dx \;=\; \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA\;.$$

Mathematically this is the same theorem as the tangential form of Green's theorem — all we have done is to juggle the symbols M and N around, changing the sign of one of them. What is different is the physical interpretation. The left side represents the flux of $\mathbf{F}$ across the closed curve C. What does the right side represent?

2. The two-dimensional divergence.

Once again, let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$. We give a name to and a notation for the integrand of the double integral on the right of (2):

$$(3) \qquad \operatorname{div}\mathbf{F} \;=\; \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}, \qquad \text{the } \textbf{divergence} \text{ of } \mathbf{F}\;.$$
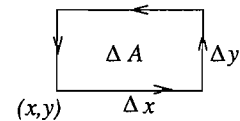
Evidently $\operatorname{div}\mathbf{F}$ is a scalar function of two variables. To get at its physical meaning, look at the small rectangle pictured. If $\mathbf{F}$ is continuously differentiable, then $\operatorname{div}\mathbf{F}$ is a continuous function, which is therefore approximately constant if the rectangle is small enough. We apply (2) to the rectangle; the double integral is approximated by a product, since the integrand is approximately constant:

$$(4) \quad \text{flux across sides of rectangle} \;\approx\; \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) \Delta A\;, \qquad \Delta A \;=\; \text{area of rectangle.}$$

Because of the importance of this approximate relation, we give a more direct derivation of it which doesn't use Green's theorem. The reasoning which follows is widely used in mathematical modeling of physical problems.

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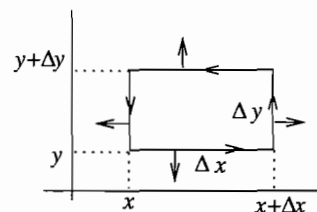Consider the small rectangle shown. We calculate an approximate value for the flux over each of its sides.

$$\text{flux across top} \approx (\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j})\,\Delta x = N(x, y + \Delta y)\,\Delta x$$
$$\text{flux across bottom} \approx (\mathbf{F}(x, y) \cdot -\mathbf{j})\Delta x = -N(x, y)\Delta x\ ;$$

adding these up,

$$\begin{matrix}\text{total flux across} \\ \text{top and bottom}\end{matrix} \approx \big(N(x, y + \Delta y) - N(x, y)\big)\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x.$$

By similar reasoning applied to the two sides,

$$\begin{matrix}\text{total flux across} \\ \text{left and right sides}\end{matrix} \approx \left(M(x + \Delta x, y) - M(x, y)\right)\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y\ .$$

Adding up the flux over the four sides, we get (4) again:

$$\begin{matrix}\text{total flux over four} \\ \text{sides of the rectangle}\end{matrix} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\,\Delta y\ .$$

Continuing our search for a physical meaning for the divergence, if the total flux over the sides of the small rectangle is positive, this means there is a net flow *out* of the rectangle. According to conservation of matter, the only way this can happen is if there is a *source* adding fluid directly to the rectangle. If the flow is taking place in a shallow tank of uniform depth, such a source can be visualized as someone standing over the tank, pouring fluid directly into the rectangle. Similarly, a net flow *into* the rectangle implies there is a *sink* withdrawing fluid from the rectangle. It is best to think of such a sink as a "negative source". The net rate (positive or negative) at which fluid is added directly to the rectangle from above may be called the "source rate" for the rectangle. Thus, since matter is conserved,

$$\text{flux over sides of rectangle} = \text{source rate for the rectangle;}$$

combining this with (4) shows that

(5) $$\text{source rate for the rectangle} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta A.$$

We now divide by ΔA and pass to the limit, getting by definition

(6) $$\text{the source rate at } (x, y) = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) = \text{div } \mathbf{F}\ .$$

The definition of the double integral as the limit of a sum shows in the usual way now that

(7) $$\text{source rate for } R = \iint_R \text{div } \mathbf{F}\, dA\ .$$

These two relations (6) and (7) interpret the divergence physically, for a flow field, and they interpret also Green's theorem in the normal form:

$$\text{total flux across } C = \text{source rate for } R$$
$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

Since Green's theorem is a mathematical theorem, one might think we have "proved" the law of conservation of matter. This is not so, since this law was needed for our interpretation of div $\mathbf{F}$ as the source rate at (x, y).

We give side-by-side the two forms of Green's theorem, first in the vector form, then in the differential form used when calculations are to be done.

<table>
<tr><td align="center">**Tangential form**</td><td align="center">**Normal form**</td></tr>
</table>

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_R \operatorname{curl} \mathbf{F}\, dA \qquad\qquad \oint_C \mathbf{F} \cdot \mathbf{n}\, ds \;=\; \iint_R \operatorname{div} \mathbf{F}\, dA$$

$$\oint_C M\, dx + N\, dy \;=\; \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \qquad \oint_C M\, dy - N\, dx \;=\; \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA$$

<table>
<tr><td align="center">work by $\mathbf{F}$</td><td align="center">flux of $\mathbf{F}$</td><td align="center">source rate</td></tr>
<tr><td align="center">around C</td><td align="center">across C</td><td align="center">for R</td></tr>
</table>

3. An interpretation for curl F.

The function curl $\mathbf{F}$ can be thought of as measuring the rotational tendency of the vector field: either as a force field or a velocity field, $\mathbf{F}$ will make a test object placed at a point P_0 spin about a vertical axis (i.e., one in the $\mathbf{k}$-direction), and the angular velocity of the spin will be proportional to $(\operatorname{curl} \mathbf{F})_0$.

To see this for the velocity field $\mathbf{v}$ of a flowing liquid, place a paddle wheel of radius a so its center is at (x_0, y_0), and its axis is vertical. We ask how rapidly the flow spins the wheel.)

If the wheel had only one blade, the velocity of the blade would be $\mathbf{F} \cdot \mathbf{t}$, the component of the flow velocity vector $\mathbf{F}$ perpendicular to the blade, i.e., tangent to the circle of radius a traced out by the blade.

paddlewheel   top view
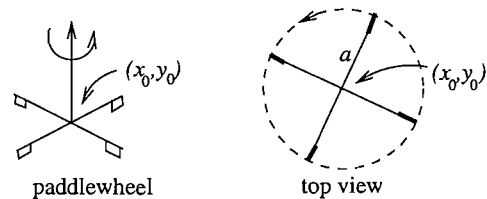
Since $\mathbf{F} \cdot \mathbf{t}$ is not constant along this circle, if the wheel had only one blade it would spin around at an uneven rate. But if the wheel has many blades, this unevenness will be averaged out, and it will spin around at approximately the *average value* of the tangential velocity $\mathbf{F} \cdot \mathbf{t}$ over the circle. Like the average value of any function defined along a curve, this average tangential velocity can be found by integrating $\mathbf{F} \cdot \mathbf{t}$ over the circle, and dividing by the length of the circle. Thus,

$$\text{speed of blade} \;=\; \frac{1}{2\pi a} \oint_C \mathbf{F} \cdot \mathbf{t}\, ds \;=\; \frac{1}{2\pi a} \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$=\; \frac{1}{2\pi a} \iint_R (\operatorname{curl} \mathbf{F})_0\, dx\, dy, \qquad \text{by Green's theorem,}$$

$$(8) \qquad\qquad \approx\; \frac{1}{2\pi a} (\operatorname{curl} \mathbf{F})_0 \pi a^2 \;,$$

where $(\operatorname{curl} \mathbf{F})_0$ is the value of the function curl $\mathbf{F}$ at (x_0, y_0). The justification for the last approximation is that if the circle formed by the paddlewheel is small, then curl $\mathbf{F}$ has approximately the value $(\operatorname{curl} \mathbf{F})_0$ over the interior R of the circle, so that multiplying this constant value by the area πa^2 of R should give approximately the value of the double integral.

From (8) we get for the tangential speed of the paddlewheel:

$$(9) \qquad\qquad \text{tangential speed} \quad \approx \quad \frac{a}{2}\,(\text{curl } \mathbf{F})_0 \ .$$

We can get rid of the a by using the angular velocity ω_0 of the paddlewheel; since the tangential speed is $a\omega_0$, (9) becomes

$$(10) \qquad\qquad\qquad\qquad\qquad \omega_0 \quad \approx \quad \frac{1}{2}\,(\text{curl } \mathbf{F})_0 \ .$$

As the radius of the paddlewheel gets smaller, the approximation becomes more exact, and passing to the limit as $a \to 0$, we conclude that, for a two-dimensional velocity field $\mathbf{F}$,

$$(11) \qquad \boxed{\text{curl } \mathbf{F} = \textit{twice the angular velocity of an infinitesimal paddlewheel at } (x,y) \ .}$$

The curl thus measures the "vorticity" of the fluid flow — its tendency to produce rotation.

A consideration of curl $\mathbf{F}$ for a force field would be similar, interpreting $\mathbf{F}$ as exerting a torque on a spinnable object — a little dumbbell with two unit masses for a gravitational field, or with two unit positive charges for an electrostatic force field.

Example 1. Calculate and interpret curl $\mathbf{F}$ for   (a) $x\,\mathbf{i} + y\,\mathbf{j}$ (b) $\omega(-y\,\mathbf{i} + x\,\mathbf{j})$

Solution. (a) curl $\mathbf{F} = 0$; this makes sense since the field is radially outward and radially symmetric, there is no favored angular direction in which the paddlewheel could spin.

(b) curl $\mathbf{F} = 2\omega$ at every point. Since this field represents a fluid rotating about the origin with constant angular velocity ω (see section V1), it is at least clear that curl $\mathbf{F}$ should be 2ω at the origin; it's not so clear that it should have this same value everywhere, but it is true.

Exercises: Section 4F