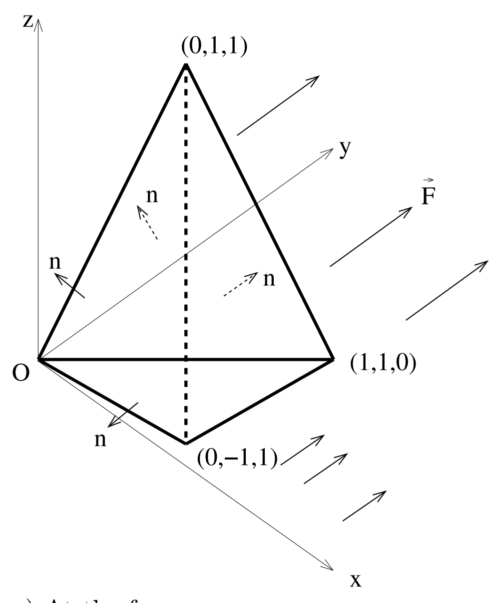
18.02 Problem Set 11 - Solutions of Part B

Problem 1

The total flux is $2\pi^2$.

Along the cylinder $x^2 + y^2 = 1$, we have $r = x^2 + y^2 = 1$ and $\hat{\mathbf{n}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$.
So $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = \dfrac{x^2+y^2}{x^2+y^2+z^2} = \dfrac{1}{1+z^2}$ and $\mathrm{dS} = r\mathrm{d}\theta\mathrm{d}z = \mathrm{d}\theta\mathrm{d}z$.
Hence, $\displaystyle\iint_S \overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}}\ \mathrm{dS} = \int_{-\infty}^{+\infty}\int_0^{2\pi} \frac{1}{1+z^2}\mathrm{d}\theta\mathrm{d}z = \int_{-\infty}^{+\infty} \frac{2\pi}{1+z^2}\mathrm{d}z =$
$= \Big[2\pi\arctan(z)\Big]_{-\infty}^{+\infty} = 2\pi^2.$

Problem 2



a) At the face:

$(\mathbf{P_0P_1P_2})$ The normal $\hat{\mathbf{n}} = -\hat{\boldsymbol{\imath}}$ and $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$.

$(\mathbf{P_0P_1P_3})$ The normal $\hat{\mathbf{n}} = \dfrac{\langle 1, -1, 1\rangle}{\sqrt{3}}$ and so $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = -\sqrt{3}x \leq 0$.

In fact, a vector perpendicular to the face and pointing outwards is obtained as $\langle 1,1,0\rangle \times \langle 0,1,1\rangle = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 1,-1,1\rangle$.

$(\mathbf{P_0P_2P_3})$ The face is obtained from the face $(\mathbf{P_0P_1P_3})$ by reflecting with respect to the xy-plane (that is, $(x,y,z) \mapsto (x,y,-z)$). So $\hat{\mathbf{n}} = \dfrac{\langle 1,-1,-1\rangle}{\sqrt{3}}$ and so $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = -\sqrt{3}x \le 0$.

$(\mathbf{P_1P_2P_3})$ The normal is $\hat{\mathbf{n}} = \hat{\boldsymbol{\jmath}}$ and $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = x \ge 0$.

b) The total flux is 0.

The flux through the single faces is:

$(\mathbf{P_0P_1P_2})$ Zero, because $\overrightarrow{\mathbf{F}} \cdot \hat{\mathbf{n}} = 0$.

$(\mathbf{P_0P_1P_3})$ The face in on the plane $x - y + z = 0$, so $\mathrm{d}\vec{\mathrm{S}} = \langle 1,-1,1\rangle \mathrm{d}x\mathrm{d}y$ and $\overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = -x\mathrm{d}x\mathrm{d}y$. Integrating over the shadow on the xy-plane, we obtain $\displaystyle\int_{\text{face}} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = \int_0^1 \int_x^1 -x \ \mathrm{d}y\mathrm{d}x = \int_0^1 x(x-1)\ \mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = -\frac{1}{6}$.

$(\mathbf{P_0P_2P_3})$ The flux is the same as for the face $(\mathbf{P_0P_1P_3})$, that is $-\dfrac{1}{6}$, because of the symmetry discussed in (a).

$(\mathbf{P_1P_2P_3})$ The face is parallel to the xz-plane, so $\mathrm{dS} = \mathrm{d}x\mathrm{d}z$.
Moreover, the face and $\overrightarrow{\mathbf{F}}$ are invariant under reflection with respect to the xy-plane. So we can integrate only on the half with $z > 0$ and the multiply by 2.
The flux is $\displaystyle 2\int_0^1 \int_0^{1-z} x\mathrm{d}x\mathrm{d}z = 2\int_0^1 \frac{(1-z)^2}{2}\mathrm{d}z = \left[\frac{(z-1)^3}{3}\right]_0^1 = \frac{1}{3}$.

Hence the total flux is $-\dfrac{1}{6} - \dfrac{1}{6} + \dfrac{1}{3} = 0$.

c) div $\overrightarrow{\mathbf{F}} = 0$, so the total flux of $\overrightarrow{\mathbf{F}}$ outgoing from the tetrahedron is 0.

Problem 3

The solid R is a cone with vertex in $(0,0,10)$ and base on the xy-plane equal to the disc of radius 10 centered at the origin.
We must show that $\iiint_R \operatorname{div} \overrightarrow{\mathbf{F}}\ \mathrm{dV} = \iint_{\partial R} \overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}}$, where ∂R is the boundary of R (in our case, the base and the lateral surface of the cone).

(LHS) $\operatorname{div} \overrightarrow{\mathbf{F}} = 2$, so $\iiint_R \operatorname{div} \overrightarrow{\mathbf{F}}\ \mathrm{dV} = 2 \cdot \mathrm{Vol}(R) = 2\frac{\pi 10^2 \cdot 10}{3} = \frac{2000\pi}{3}$.

(RHS) The unit normal vector outgoing from the base is $-\hat{\boldsymbol{k}}$ and $\overrightarrow{\mathbf{F}} \cdot (-\hat{\boldsymbol{k}}) = 0$, so the flux through the base is 0.
The lateral surface is given by $z = f(x,y) = 10 - \sqrt{x^2+y^2}$, so
$\mathrm{d}\vec{\mathrm{S}} = \langle -f_x, -f_y, 1\rangle\, \mathrm{d}x\mathrm{d}y = \left\langle \frac{x}{r}, \frac{y}{r}, 1 \right\rangle r\mathrm{d}r\mathrm{d}\theta$ (we switched to polar coordinates) and $\overrightarrow{\mathbf{F}} \cdot \mathrm{d}\vec{\mathrm{S}} = r^2 \mathrm{d}r\mathrm{d}\theta$.
Hence, the flux is $\int_0^{2\pi}\int_0^{10} r^2 \mathrm{d}r\mathrm{d}\theta = 2\pi \left[\frac{r^3}{3}\right]_0^{10} = \frac{2000\pi}{3}$.

Problem 4

a) $\iint_S (f\nabla g)\cdot \hat{\mathbf{n}}\ \mathrm{dS} = \iiint_D \operatorname{div}(f\nabla g)\,\mathrm{dV}$.
On the LHS, $f\nabla g \cdot \hat{\mathbf{n}} = f\frac{\partial g}{\partial n}$.
On the RHS, $\operatorname{div}\,(f\nabla g) = \operatorname{div}\left(fg_x\hat{\boldsymbol{\imath}} + fg_y\hat{\boldsymbol{\jmath}} + fg_z\hat{\boldsymbol{k}}\right) =$
$= (fg_x)_x + (fg_y)_y + (fg_z)_z = f_xg_x + fg_{xx} + f_yg_y + fg_{yy} + f_zg_z + fg_{zz} =$
$= (f_xg_x + f_yg_y + f_zg_z) + f(g_{xx} + g_{yy} + g_{zz}) = \nabla f \cdot \nabla g + f\nabla^2 g.$

b) If $f = 1$ and $g = u$ is harmonic, then $\nabla f = 0$ and $\nabla^2 g = 0$, so
$\nabla f \cdot \nabla g + f\nabla^2 g = 0$.
Hence, Green's first identity gives $\iint_S \frac{\partial u}{\partial n}\mathrm{dS} = 0$.

c) $\iint_S f\frac{\partial g}{\partial n}\mathrm{dS} = \iiint_D \left(\nabla f \cdot \nabla g + f\nabla^2 g\right)\mathrm{dV}$
$\iint_S g\frac{\partial f}{\partial n}\mathrm{dS} = \iiint_D \left(\nabla g \cdot \nabla f + g\nabla^2 f\right)\mathrm{dV}$

Subtracting the second row from the first row we obtain

$$\iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) \mathrm{dS} = \iiint_D \left(f\nabla^2 g - g\nabla^2 f \right) \mathrm{dV}.$$

d) $\nabla^2 v = 0$ (that is, $v = \dfrac{1}{\rho}$ is harmonic) outside the origin.

In fact, $v_x = -\dfrac{1}{\rho^2}\rho_x = -\dfrac{x}{\rho^3}$ and $v_{xx} = \left(-\dfrac{x}{\rho^3}\right)_x = -\dfrac{1}{\rho^3} + \dfrac{3x}{\rho^4}\dfrac{x}{\rho} = -\dfrac{1}{\rho^3} + \dfrac{3x^2}{\rho^5}$.

Similarly, $v_{yy} = -\dfrac{1}{\rho^3} + \dfrac{3y^2}{\rho^5}$ and $v_{zz} = -\dfrac{1}{\rho^3} + \dfrac{3z^2}{\rho^5}$, so that

$\nabla^2 v = -\dfrac{3}{\rho^3} + \dfrac{3x^2 + 3y^2 + 3z^2}{\rho^5} = 0.$

Let's apply Green's second identity to u and v, with D equal to the region $a < \rho < b$ and S equal to the union of the two spheres S_a and S_b.

$\displaystyle\iint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{dS} = \iiint_D \left(u\nabla^2 v - v\nabla^2 u \right) \mathrm{dV}.$

The RHS is zero, because $\nabla^2 u = \nabla^2 v = 0$, so we get

$\displaystyle\iint_{S_a} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{dS} + \iint_{S_b} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{dS} = 0.$

Along S_b, $v = 1/b$, so $\displaystyle\iint_{S_b} -v\frac{\partial u}{\partial n}\mathrm{dS} = -\frac{1}{b}\iint_{S_b} \frac{\partial u}{\partial n}\mathrm{dS} = 0$ because of (b).

Similarly, $\displaystyle\iint_{S_a} -v\frac{\partial u}{\partial n}\mathrm{dS} = 0.$

The normal vector on S_b outgoing from the region D is $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}} = \dfrac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}{b}$,

so $\dfrac{\partial v}{\partial n} = \dfrac{\partial v}{\partial \rho} = -\dfrac{1}{\rho^2} = -\dfrac{1}{b^2}$ along S_b.

The normal vector on S_a outgoing from the region D is $\hat{\mathbf{n}} = -\hat{\boldsymbol{\rho}} = -\dfrac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}{a}$,

so $\dfrac{\partial v}{\partial n} = -\dfrac{\partial v}{\partial \rho} = \dfrac{1}{\rho^2} = \dfrac{1}{a^2}$ along S_a.

Therefore, $\displaystyle\frac{1}{a^2}\iint_{S_a} u \ \mathrm{dS} = \frac{1}{b^2}\iint_{S_b} u \ \mathrm{dS}.$

e) Let $b > 0$. We want to show that $\displaystyle\frac{1}{4\pi b^2}\iint_{S_b} w \ \mathrm{dS} = w(\mathbf{0})$, where S_b is the sphere of radius b centered at the origin $\mathbf{0}$.

Using (d), we have $\displaystyle\frac{1}{4\pi b^2}\iint_{S_b} w \ \mathrm{dS} = \frac{1}{4\pi a^2}\iint_{S_a} w \ \mathrm{dS}$ for every $a > 0$.

In particular, $\displaystyle\frac{1}{4\pi b^2}\iint_{S_b} w \ \mathrm{dS} = \lim_{a\to 0}\frac{1}{4\pi a^2}\iint_{S_a} w \ \mathrm{dS} = w(\mathbf{0}).$

To deduce the Mean Value Theorem for the point P, we must choose D to be given by the locus of points whose distance from P is between a and b (that is, the region enclosed by two spheres of radii a and b centered at P) and $v(x, y, z)$ to be the function given by $v(Q) = \dfrac{1}{|\overrightarrow{PQ}|}$.