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Recall that yesterday we saw, no, two days ago we learned about the curl of a vector field in space.

And we said the curl of F is defined by taking a cross product between the symbol dell and the vector F.

Concretely, the way we would compute this would be by putting the components of F into this determinant and expanding and then getting a vector with components Ry minus Qz, Pz minus Rx and Qx minus Py. I think I also tried to explain very quickly what the significance of a curl is.

Just to tell you again very quickly, basically curl measures, if you mention that your vector field measures the velocity in some fluid then the curl measures how much rotation is taking place in that fluid.

Measures the rotation part of a velocity field.

More precisely the direction corresponds to the axis of rotation and the magnitude corresponds to twice the angular velocity.

Just to give you a few quick examples.

If I take a constant vector field, so everything translates at the same speed, then obviously when you take the partial derivatives you will just get a bunch of zeros so the curl will be zero.

If you take a vector field that stretches things, let's say, for example, we are going to stretch things along the x-axis, that would be a vector field that goes parallel to the x direction but maybe, say, x times i. So that when you are in front of a plane of a blackboard you are moving forward, when you are behind you are moving backwards, things are getting expanded in the x direction.

If you compute the curl, you can check each of these.

Again, they are going to be zero.

There is no curl. This is not what curl measures.

I mean, actually, what measures expansion, stretching is actually divergence.

If you take the divergence of this field, you would get one plus zero plus zero, it looks like it will be one, so in case you don't remember, I mean divergence precisely measures this stretching effect in your field. And, on the other hand, if you take something that corresponds to, say, rotation about the z-axis at unit angular velocity -- That means they are going to moving in circles around the z-axis. One way to write down this field, let's see, the z component is zero because everything is moving horizontally.

And in the x and y directions, if you look at it from above, well, it is just going to be our good old friend the vector field that rotates everything [at unit speed?].

And we have seen the formula for this one many times.

The first component is minus y, the second one is x.

Now, if you compute the curl of this guy, you will get zero, zero, two, two k. And so k is the axis of rotation, two is twice the angular velocity.

And now, of course, you can imagine much more complicated motions where you will have -- For example, if you look at the Charles River very carefully then you will see that water is flowing, generally speaking, towards the ocean. But, at the same time, there are actually a few eddies in there and with water swirling. Those are the places where there is actually curl in the flow.

Yes. I don't know how to turn out the lights a bit, but I'm sure there is a way.

Does this do it? Is it working?

OK. You're welcome.

Hopefully it is easier to see now.

That was about curl. Now, why do we care about curl besides this motivation of understanding motions?

One place where it comes up is when we try to understand whether a vector field is conservative.

Remember we have seen that a vector field is conservative if and only if its curl is zero. That is the situation in which we are allowed to try to look for a potential function and then use the fundamental theorem.

But another place where this comes up, if you remember what we did in the plane, curl also came up when we tried to convert nine integrals into double integrals. That was Greene's theorem.

Well, it turns out we can do the same thing in space and that is called Stokes' theorem. What does Stokes' theorem say?

It says that the work done by a vector field along a closed curve can be replaced by a double integral of curl F.

Let me write it using the dell notation.

That is curl F. Dot ndS on a suitably chosen surface. That is a very strange kind of statement. But actually it is not much more strange than things we have seen before.

I should clarify what this means.

C has to be a closed curve in space.

And S can be any surface bounded by C.

For example, what Stokes' theorem tells me is that let us say that I have to compute some line integral on maybe, say, the unit circle in the x, y plane. Of course I can set a line integral directly and compute it by setting x equals cosine T, y equals sine T, z equals zero.

But maybe sometimes I don't want to do that because my vector field is really complicated.

And instead I will want to reduce things to a surface integral. Now, I know that you guys are not necessarily fond of computing flux of vector fields for surfaces so maybe you don't really see the point.

But sometimes it is useful. Sometimes it is also useful backwards because, actually, you have a surface integral that you would like to turn into a line integral.

What Stokes' theorem says is that I can choose my favorite surface whose boundary is this circle.

I could choose, for example, a half sphere if I want or I can choose, let's call that s1, I don't know, a pointy thing, s2. Probably the most logical one, actually, would be just to choose a disk in the x, y plane. That would probably be the easiest one to set up for calculating flux.

Anyway, what Stokes' theorem tells me is I can choose any of these surfaces, whichever one I want, and I can compute the flux of curl F through this surface. Curl F is a new vector field when you have this formula that gives you a vector field you compute its flux through your favorite surface, and you should get the same thing as if you had done the line integral for F. That is the statement.

Now, there is a catch here. What is the catch?

Well, the catch is we have to figure out what conventions to use because remember when we have a surface there are two possible orientations. We have to decide which way we will counter flux positively, which way we will counter flux negatively. And, if we change our choice, then of course the flux will become the opposite.

Well, similarly to define the work, I need to choose which way I am going to run my curve. If I change which way I go around the curve then my work will become the opposite.

What happens is I have to orient the curve C and the surface S in compatible ways. We have to figure out what the rule is for how the orientation of S and that of C relate to each other. What about orientation?

Well, we need the orientations of C and S to be compatible and they have to explain to you what the rule is.

Let me show you a picture. The rule is if I walk along C with S to my left then the normal vector is pointing up for me. Let me write that.

If I walk along C, I should say in the positive direction, in the direction that I have chosen to orient C.

With S to my left then n is pointing up for me.

Here is the example. If I am walking on this curve, it looks like the surface is to my left.

And so the normal vector is going towards what is up for me.

Any questions about that? I see some people using their right hands. That is also right-handable which I am going to say in just a few moments.

That is another way to remember this.

Before I tell you about the right-handable version, let me just try something. Actually, I am not happy with this orientation of C and I want to orient my curve C going clockwise on the picture. So the other orientation.

Then, if I walk on it this way, the surface would be to my right. You can remember, if it helps you, that if a surface is to your right then the normal vector will go down.

The other way to think about this rule is enough because if you are walking clockwise, well, you can change that to counterclockwise just by walking upside down.

This guy is walking clockwise on C.

And while for him, if you look carefully at the picture, the surface is actually to his left when you flip upside down. Yeah, it is kind of confusing.

But, anyway, maybe it's easier if you actually rotate in the picture. And now it is getting actually really confusing because his walking upside up with, actually, the surface is to his left.

I mean where he is at here is actually at the front and this is the back, but that is kind of hard to see.

Anyway, whichever method will work best for you.

Perhaps it is easiest to first do it with the other orientation, this one, and this side, if you want the opposite one, then you will just flip everything.

Now, what is the other way of remembering this with the right-hand rule? First of all, take your right hand, not your left.

Even if your right hand is actually using a pen or something like that in your right hand do this.

And let's take your fingers in order.

First your thumb. Let's make your thumb go along the object that has only one dimension in there.

That is the curve. Well, let's look at the top picture up there. I want my thumb to go along the curve so that is kind of towards the right.

Then I want to make my index finger point towards the surface. Towards the surface I mean towards the interior of the surface from the curve.

And when I am on the curve I am on the boundary of the surface, so there is a direction along the surface that is the curve and the other one is pointing into the surface.

That one would be pointing kind of to the back slightly up maybe, so like that. And now your middle finger is going to point in the direction of the normal vector.

That is up, at least if you have the same kind of right hand as I do. The other way of doing it is using the right-hand rule along C positively.

The index finger towards the interior of S.

Sorry, I shouldn't say interior. I should say tangent to S towards the interior of S. What I mean by that is really the part of S that is not its boundary, so the rest of the surface. Then the middle finger points parallel to n. Let's practice.

Let's say that I gave you this curve bounding this surface.

Which way do you think the normal vector will be going?

Up. Yes. Everyone is voting up. Imaging that I am walking around C. That is to my left.

Normal vector points up. Imagine that you put your thumb along C, your index towards S and then your middle finger points up. Very good.

N points up. Another one.

It is interesting to watch you guys.

I think mostly it is going up. The correct answer is it goes up and into the cone. How do we see that?

Well, one way to think about it is imagine that you are walking on C, on the rim of this cone. You have two options.

Imagine that you are walking kind of inside or imagine that you are walking kind of outside. If you are walking outside then S is to your right, but it does not sound good.

Let's say instead that you are walking on the inside of a cone following the boundary. Well, then the surface is to your left. And so the normal vector will be up for you which means it will be pointing slightly up and into the cone. Another way to think about it, through the right-hand rule, from this way index going kind of down because the surface goes down and a bit to the back.

And then the normal vector points up and in.

Yet another way, if you deform continuously your surface then the conventions will not change.

See, this is kind of [UNINTELLIGIBLE] in a way. You can deform things and nothing will change. So what if we somehow flatten our cone, push it a bit up so that it becomes completely flat?

Then, if you had a flat disk with the curve going counterclockwise, the normal vector would go up.

Now take your disk with its normal vector sticking up.

If you want to paint the face a different color so that you can remember that was beside with a normal vector and then push it back down to the cone, you will see that the painted face, the one with the normal vector on that side is the one that is inside and up.

Does that make sense? Anyway, I think you have just to play with these examples for long enough and get it.

OK. The last one. Let's say that I have a cylinder. So now this guy has actually two boundary curves, C and C prime.

And let's say I want to orient my cylinder so that the normal vector sticks out. How should I choose the orientation of my curves? Let's start with, say, the bottom one. Would the bottom one be going clockwise or counterclockwise. Most people seem to say counterclockwise, and I agree with that.

Let me write that down and claim C prime should go counterclockwise. One way to think about it, actually, it's quite easy, you mentioned that you're walking on the outside of the cylinder along C prime.

If you want to walk along C prime so that the cylinder is to your left, that means you have to actually go counterclockwise around it. The other way is use your right hand. Say when you're at the front of C prime, your thumb points to the right, your index points up because that's where the surface is, and then your middle finger will point out.

What about C? Well, C I claim we should be doing clockwise. I mean think about just walking again on the surface of the cylinder along C.

If you walk clockwise, you will see that the surface is to your left or use the right-hand rule.

Now, if a problem gives you neither the orientation of a curve nor that of the surface then it's up to you to make them up. But you have to make them up in a consistent way. You cannot choose them both at random. All right.

Now we're all set to try to use Stokes' theorem.

Well, let me do an example first.

The first example that I will do is actually a comparison.

Stokes' versus Green. I want to show you how Green's theorem for work that we saw in the plane, but also involved work and curl and so on, is actually a special case of this.

Let's say that we will look at the special case where our curve C is actually a curve in the x, y plane.

And let's make it go counterclockwise in the x, y plane because that's what we did for Green's theorem.

Now let's choose a surface bounded by this curve.

Well, as I said, I could make up any surface that comes to my mind. But, if I want to relate to this stuff, I should probably stay in the x, y plane. So I am just going to take my surface to be the piece of the x, y plane that is inside my curve. So let's say S is going to be a portion of x, y plane bounded by a curve C, and the curve C goes counterclockwise.

Well, then I should look at [the table?].

For work along C of my favorite vector field F dot dr.

So that will be the line integral of Pdx plus Qdy.

Like I said, if I call the components of my field P, Q and R, it will be Pdx plus Qdy plus Rdz, but I don't have any Z here.

Dz is zero on C. If I evaluate for line integral, I don't have any term involving dz.

Z is zero. Now, let's see what Strokes says. Stokes says instead I can compute for flux through S of curve F.

But now what's the normal vector to my surface?

Well, it's going to be either k or negative k.

I just have to figure out which one it is.

Well, if you followed what we've done there, you know that the normal vector compatible with this choice for the curve C is the one that points up.

My normal vector is just going to be k hat, so I am going to replace my normal vector by k hat.

That means, actually, I will be integrating curl dot k. That means I am integrating the z component of curl. Let's look at curl F dot k.

That's the z component of curl F.

And what's the z component of curl?

Well, I conveniently still have the values up there.

It's Q sub x minus P sub y. My double integral becomes double integral of Q sub x minus P sub y.

What about dS? Well, I am in a piece of the x, y plane, so dS is just dxdy or your favorite combination that does the same thing. Now, see, if you look at this equality, integral of Pdx plus Qdy along a closed curve equals double integral of Qx minus Py dxdy.

That is exactly the statement of Green's theorem.

I mean except at that time we called things m and n, but really that shouldn't matter.

This tells you that, in fact, Green's theorem is just a special case of Stokes' in the x, y plane.

Now, another small remark I want to make right away before I forget, you might think that these rules that we've made up about compatibility of orientations are completely arbitrary. Well, they are literally in the same way as our convention for which we guy curl is arbitrary.

We chose to make the curl be this thing and not the opposite which would have been pretty much just as sensible.

And, ultimately, that comes from our choice of making the cross-product be what it is but of the opposite.

Ultimately, it all comes from our preference for right-handed coordinate systems. If we had been on the planet with left-handed coordinate systems then actually our conventions would be all the other way around, but they are this way. Any other questions?

A surface that you use in Stokes' theorem is usually not going to be closed because its boundary needs to be the curve C. So if you had a closed surface you wouldn't know where to put your curve.

I mean of course you could make a tiny hole in it and get a tiny curve. Actually, what that would say, and we are going to see more about that so not very important right now, but what we would see is that for a close surface we would end up getting zero for the flux.

And that is actually because divergence of curl is zero, but I am getting ahead of myself.

We are going to see that probably tomorrow in more detail. Stokes' theorem only works if you can make sense of this. That means you need your vector field to be continuous and differentiable everywhere on the surface S. Now, why is that relevant?

Well, say that your vector field was not defined at the origin and say that you wanted to do, you know, the example that I had first with the unit circling the x, y plane. Normally, the most sensible choice of surface to apply Stokes' theorem to would be just the flat disk in the x, y plane.

But that assumes that your vector field is well-defined there. If your vector field is not defined at the origin but defined everywhere else you cannot use this guy, but maybe you can still use, say, the half-sphere, for example.

Or, you could use a piece of cylinder plus a flat top or whatever you want but not pressing for the origin.

So you could still use Stokes but you'd have to be careful about which surface you choose. Now, if instead your vector field is not defined anywhere on the z-axis then you're out of luck because there is no way to find a surface bounded by this unit circle without crossing the z-axis somewhere.

Then you wouldn't be able to Stokes' theorem at all or at least not directly. Maybe I should write it F defines a differentiable everywhere on this.

But we don't care about what happens outside of this.

It's really only on the surface that we need it to be OK.

I mean, again, 99% of the vector fields that we see in this class are defined everywhere so that's not an urgent concern, but still.

OK. Should we move on?

Yes. I have a yes. Let me explain to you quickly why Stokes is true. How do we prove a theorem like that? Well, the strategy, I mean there are other ways, but the least painful strategy at this point is to observe what we already know is a special case of Stokes's theorem.

Namely we know the case where the curve is actually in the x, y plane and the surface is a flat piece of the x, y plane because that's Green's theorem which we proved a while ago. We know it for C and S in the x, y plane. Now, what if C and S were, say, in the y, z plane instead of the x, y plane? Well, then it will not quite give the same picture because the normal vector would be i hat instead of k hat and they would be having different notations and it would be integrating with y and z.

But you see that it would become, again, exactly the same formula. We'd know it for any of the coordinate planes. In fact, I claim we know it for absolutely any plane. And the reason for that is, sure, when we write it in coordinates, when we write that this line integral is integral of Pdx plus Qdy plus Rdz or when we write that the curl is given by this formula we use the x, y, z coordinate system.

But there is something I haven't quite told you about.

Which is if I switch to any other right-handed coordinate system, so I do some sort of rotation of my space coordinates, then somehow the line integral, the flux integral, the notion of curl makes sense in coordinates. And the reason is that they all have geometric interpretations. For example, when I think of this as the work done by a force, well, the force doesn't care whether it's being put in x, y coordinates this way or that way.

It still does the same work because it's the same force.

And when I say that the curl measures the rotation in a motion, well, that depends on which coordinates you use. And the same for interpretation of flux. In fact, if I rotated my coordinates to fit with any other plane, I could still do the same things. What I'm trying to say is, in fact, if C and S are in any plane then we can still claim that it reduces to Green's theorem.

It will be Green's theorem not in x, y, z coordinates but in some funny rotated coordinate systems.

What I'm saying is that work, flux and curl makes sense independently of coordinates.

Now, this has to stop somewhere. I can start claiming that I can somehow bend my coordinates to a plane, any surface is flat.

That doesn't really work. But what I can say is if I have any surface I can cut it into tiny pieces.

And these tiny pieces are basically flat.

So that's basically the idea of a proof.

I am going to decompose my surface into very small flat pieces. Given any S we are just going to decompose it into tiny almost flat pieces.

For example, if I have my surface like this, what I will do is I will just cut it into tiles.

I mean a good example of that is if you look at [UNINTELLIGIBLE], for example, it's made of all these hexagons and pentagons.

Well, actually, they're not quite flat in the usual rule, but you could make them flat and it would still look pretty much like a sphere. Anyway, you're going to cut your surface into lots of tiny pieces.

And then you can use Stokes' theorem on each small piece.

What it says on each small flat piece -- It says that the line integral along say, for example, this curve is equal to the flux of a curl through this tiny piece of surface. And now I will add all of these terms together. If I add all of the small contributions to flux I get the total flux.

What if I add all of the small line integrals?

Well, I get lots of extra junk because I never asked to compute the line integral along this. But this guy will come in twice when I do this little plate and when I do that little plate with opposite orientations. When I sum all of the little line integrals together, all of the inner things cancel out, and the only ones that I go through only once are those that are at the outer most edges.

So, when I sum all of my works together, I will get the work done just along the outer boundary C.

Sum of work around each little piece is just actually the work along C, the outer curve. And the sum of the flux for each piece is going to be the flux through S.

From Stokes' theorem for flat surfaces, I can get it for any surface. I am cheating a little bit because you would actually have to check carefully that this approximately where you flatten the little pieces that are almost flat is [UNINTELLIGIBLE]. But, trust me,

it actually works.

Let's do an actual example. I mean I said example, but that was more like getting us ready for the proof so probably that doesn't count as an actual example.

I should probably keep these statements for now so I am not going to erase this side.

Let's do an example. Let's try to find the work of vector field zi plus xj plus yk around the unit circle in the x, y plane counterclockwise. The picture is conveniently already there. Just as a quick review, let's see how we do that directly.

If we do that directly, I have to find the integral along C. So F dot dr becomes zdx plus xdy plus ydz. But now we actually know that on this circle, well, z is zero.

And we can parameterize x and y, the unit circle in the x, y plane, so we can take x equals cosine t, y equals sine t. That will just become the integral over C. Well, z times dx, z is zero so we have nothing, plus x is cosine t times dy is -- Well, if y is sine t then dy is cosine tdt plus ydz but z is zero. Now, the range of values for t, well, we are going counterclockwise around the entire circle so that should go from zero to 2pi.

We will get integral from zero to 2pi of cosine square tdt which, if you do the calculation, turns out to be just pi. Now, let's instead try to use Stokes' theorem to do the calculation.

Now, of course the smart choice would be to just take the flat unit disk. I am not going to do that.

That would be too boring. Plus we have already kind of checked it because we already trust Green's theorem.

Instead, just to convince you that, yes, I can choose really any surface I want, let's say that I'm going to choose a piece of paraboloid z equals one minus x squared minus y squared.

Well, to get our conventions straight, we should take the normal vector pointing up for compatibility with our choice.

Well, we will have to compute the flux through S.

We don't really have to because we could have chosen the disk, it would be easier, but if we want to do it this way we will compute the flux of curl F through our paraboloid.

How do we do that? Well, we need to find the curl and we need to find ndS. Let's start with the curl.

Curl F let's take the cross-product between dell and F which is zxy. If we compute this, the i component will be one minus zero.

It looks like it is one i. Minus the j component is zero minus one. Plus the k component is one minus zero. In fact, the curl of the field is one, one, one. Now, what about ndS?

Well, this is a surface for which we know z is a function of x and y. ndS we can write as, let's call this F of xy, then we can use the formula that says ndS equals negative F sub x, negative F sub y, one dxdy, which here gives us 2x, 2y, one dxdy. Now, when we want to compute the flux, we will have to do double integral over S of one, one, one dot product with 2x, 2y, one dxdy.

It will become the double integral of 2x plus 2y plus one dxdy. And, of course, the region which we are integrating, the range of values of x and y will be the shadow of our surface.

That is just going to be, if you look at this paraboloid from above, all you will see is the unit disk so it will be a double integral of the unit disk.

And the way we will do that, one way is to switch to polar coordinates and do the calculation and then you will end up with pi. The other way is to try to do it by symmetry. Observe, when you integrate x above this, x is as negative on the left as it is positive on the right. So the integral of x will be zero. The integral of y will be zero also by symmetry. Then the integral of one dxdy will just be the area of this unit disk which is pi.

That was our first example. And, of course, if you're actually free to choose your favorite surface, there is absolutely no reason why you would actually choose this paraboloid in this example. I mean it would be much easier to choose a flat disk. OK.

Tomorrow I will tell you a few more things about curl fits in with conservativeness and with the divergence theorem, Stokes all together, and we will look at Practice Exam 4B so please bring the exam with you.

with you.