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So, there's the third exam in this class will be on Thursday. I'm sure you already know. Just remind you, you will go to the usual place when there's an exam. Namely, if your last name starts with letter a to r, or if you're left handed, then you should go to Walker. And if you start with s to z, then you should come here. As usual, there's no documents, no calculators allowed. And, for those of you who can't make it, then you have to talk to us in advance. That there will be make-ups next week as usual, not on Monday because Monday is a holiday. Yea! [LAUGHTER] OK, the other piece of information is that I will have slightly extended office hours because I know a lot of you are anxious about the test.

So, today I will be in my office after class until 3:30, and tomorrow from 3:30-6:00. And, well let's see. So, before we actually start reviewing for the test, I still have to tell you a few small things because I promised to say a few words about what's the difference, or precisely, what's the difference between curl being zero and a field being a gradient field, and why we have this assumption that our vector field had to be defined everywhere for a field with curl zero to actually be conservative for our test for gradient fields to be valid? So --

More about validity of Green's theorem and things like that. So, we've seen the statement of Green's theorem in two forms. Both of them have to do with comparing a line integral along a closed curve to a double integral over the region inside enclosed by the curve. So, one of them says the line integral for the work done by a vector field along a closed curve counterclockwise is equal to the double integral of a curl of a field over the enclosed region. And, the other one says the total flux out of the region, so, the flux through the curve is equal to the double integral of divergence of a field in the region.

So, in both cases, we need the vector field to be defined not only, I mean, the left hand side makes sense if a vector field is just defined on the curve because it's just a line integral on C . We don't care what happens inside. But, for the right-hand side to make sense, and therefore for the equality to make sense, we need the vector field to be defined everywhere inside the region. So, I said, if there is a point somewhere in here where my vector field is not defined, then it doesn't work. And actually, we've seen that example. Oops.

So, this only works if F and its derivatives are defined everywhere in the region, R . Otherwise, we are in trouble. OK, so we've seen for example that if I gave you the vector field $\frac{y}{x^2} \mathbf{i} - \frac{x}{y^2} \mathbf{j}$ over $x^2 + y^2 > 0$, so that's the same vector field that was on that problem set a couple of weeks ago. Then, well, f is not defined at the origin, but it's defined everywhere else. And, wherever it's defined, it's curl is zero.

I should say everywhere it's -- And so, if we have a closed curve in the plane, well, there's two situations. One is if it does not enclose the origin. Then, yes, we can apply Green's theorem and it will tell us that it's equal to the double integral in here

of $\text{curl } \mathbf{F} \cdot d\mathbf{A}$, which will be zero because this is zero. However, if I have a curve that encloses the origin, let's say like this, for example, then, well, I cannot use the same method because the vector field and its curl are not defined at the origin.

And, in fact, you know that ignoring the problem and saying, well, the curl is still zero everywhere, will give you the wrong answer because we've seen an example. We've seen that along the unit circle the total work is 2π not zero. So, we can't use Green. However, we can't use it directly. So, there is an extended version of Green's theorem that tells you the following thing. Well, it tells me that even though I can't do things for just this region enclosed by C prime, I can still do things for the region in between two different curves.

OK, so let me show you what I have in mind. So, let's say that I have my curve C' . Where's my yellow chalk? Oh, here. So, I have this curve C' . I can't apply Green's theorem inside it, but let's get out the smaller thing. So, that one I'm going to make going clockwise. You will see why. Then, I could say, well, let me change my mind. This picture is not very well prepared. That's because my writer is on strike. OK, so let's say we have C' and C'' both going counterclockwise.

Then, I claim that Green's theorem still applies, and tells me that the line integral along C prime minus the line integral along C double prime is equal to the double integral over the region in between. So here, now, it's this region with the hole of the curve. And, well, in our case, that will turn out to be zero because curl is zero. OK, so this doesn't tell us what each of these two line integrals is. But actually, it tells us that they are equal to each other. And so, by computing one, you can see actually that for this vector field, if you take any curve that goes counterclockwise around the origin, you would get two pi no matter what the curve is.

So how do you get to this? Why is this not like conceptually a new theorem? Well, just think of the following thing. I'm not going to do it on top of that because it's going to be messy if I draw too many things. But, so here I have my C'' . Here, I have C' . Let me actually make a slit that will connect them to each other like this. So now if I take, see, I can form a single closed curve that will enclose all of this region with kind of an infinitely thin slit here counterclockwise. And so, if I go counterclockwise around this region, basically I go counterclockwise along the outer curve. Then I go along the slit.

Then I go clockwise along the inside curve, then back along the slit. And then I'm done. So, if I take the line integral along this big curve consisting of all these pieces, now I can apply Green's theorem to that because it is the usual counterclockwise curve that goes around a region where my field is well-defined. See, I've eliminated the origin from the picture. And, so the total line integral for this thing is equal to the integral along C prime, I guess the outer one. Then, I also need to have what I do along the inner side. And, the inner side is going to be C double prime, but going backwards because now I'm going clockwise on C prime so that I'm going counterclockwise around the shaded region.

Well, of course there will be contributions from the line integral along this wide segment. But, I do it twice, once each way. So, they cancel out. So, the white segments cancel out. You probably shouldn't, in your notes, write down white segments because probably they are not white on your paper. But, hopefully you get the meaning of what I'm trying to say. OK, so basically that tells you, you can still play tricks with Green's theorem when the region has holes in it. You just had to be

careful and somehow subtract some other curve so that together things will work out. There is a similar thing with the divergence theorem, of course, with flux and double integral of $\text{div } f$, you can apply exactly the same argument.

OK, so basically you can apply Green's theorem for a region that has several boundary curves. You just have to be careful that the outer boundary must go counterclockwise. The inner boundary either goes clockwise, or you put a minus sign. OK, and the last cultural note, so, the definition, we say that a region in the plane, sorry, I should say a connected region in the plane, so that means -- So, connected means it consists of a single piece. OK, so, connected, there is a single piece. These two guys together are not connected. But, if I join them, then this is a connected region. We say it's simply connected --

-- if any closed curve in it, OK, so I need to give a name to my region, let's say R , any closed curve in R , bounds, no, sorry. If the interior of any closed curve in R -- -- is also contained in R . So, concretely, what does that mean? That means the region, R , does not have any holes inside it. Maybe I should draw two pictures to explain what I mean. So, this guy here is simply connected while --

-- this guy here is not simply connected because if I take this curve, that's a curve inside my region. But, the piece that it bounds is not actually entirely contained in my region. And, so why is that relevant? Well, if you know that your vector field is defined everywhere in a simply connected region, then you don't have to worry about this question of, can I apply Green's theorem to the inside? You know it's automatically OK because if you have a closed curve, then the vector field is, I mean, if a vector field is defined on the curve it will also be defined inside.

OK, so if the domain of definition -- -- of a vector field is defined and differentiable -- -- is simply connected -- -- then we can always apply -- -- Green's theorem -- -- and, of course, provided that we do it on a curve where the vector field is defined. I mean, your line integral doesn't make sense so there's nothing to compute. But, if you have, so, again, the argument would be, well, if a vector field is defined on the curve, it's also defined inside.

So, see, the problem with that vector field here is precisely that its domain of definition is not simply connected because there is a hole, namely the origin. OK, so for this guy, domain of definition, which is plane minus the origin with the origin removed is not simply connected. And so that's why you have this line integral that makes perfect sense, but you can't apply Green's theorem to it. So now, what does that mean a particular? Well, we've seen this criterion that if a curl of the vector field is zero and it's defined in the entire plane, then the vector field is conservative, and it's a gradient field.

And, the argument to prove that is basically to use Green's theorem. So, in fact, the actual optimal statement you can make is if a vector field is defined in a simply connected region, and its curl is zero, then it's a gradient field. So, let me just write that down. So, the correct statement, I mean, the previous one we've seen is also correct. But this one is somehow better and closer to what exactly is needed if curl F is zero and the domain of definition where F is defined is simply connected --

-- then F is conservative. And that means also it's a gradient field. It's the same thing. OK, any questions on this? No? OK, some good news. What I've just said here won't come up on the test on Thursday. OK. (APPLAUSE) Still, it's stuff that you

should be aware of generally speaking because it will be useful, say, on the next week's problem set. And, maybe on the final it would be, there won't be any really, really complicated things probably, but you might need to be at least vaguely aware of this issue of things being simply connected.

And by the way, I mean, this is also somehow the starting point of topology, which is the branch of math that studies the shapes of regions. So, in particular, you can try to distinguish domains in the plains by looking at whether they're simply connected or not, and what kinds of features they have in terms of how you can joint point what kinds of curves exist in them. And, since that's the branch of math in which I work, I thought I should tell you a bit about it.

OK, so now back to reviewing for the exam. So, I'm going to basically list topics. And, if time permits, I will say a few things about problems from practice exam 3B. I'm hoping that you have it or your neighbor has it, or you can somehow get it. Anyway, given time, I'm not sure how much I will say about the problems in and of themselves. OK, so the main thing to know about this exam is how to set up and evaluate double integrals and line integrals. OK, if you know how to do these two things, then you are in much better shape than if you don't.

And -- So, the first thing we've seen, just to write it down, there's two main objects. And, it's kind of important to not confuse them with each other. OK, there's double integrals of our regions of some quantity, dA , and the other one is the line integral along a curve of a vector field, $F \cdot dr$ or $F \cdot Mds$ depending on whether it's work or flux that we are trying to do. And, so we should know how to set up these things and how to evaluate them. And, roughly speaking, in this one you start by drawing a picture of the region, then deciding which way you will integrate it.

It could be $dx dy$, $dy dx$, $r dr d\theta$, and then you will set up the bound carefully by slicing it and studying how the bounds for the inner variable depend on the outer variable. So, the first topic will be setting up double integrals. And so, remember, OK, so maybe I should make this more explicit. We want to draw a picture of R and take slices in the chosen way so that we get an iterated integral. OK, so let's do just a quick example. So, if I look at problem one on the exam 3B, it says to look at the line integral from zero to one,

line integral from x to $2x$ of possibly something, but $dy dx$. And it says, let's look at how we would set this up the other way around by exchanging x and y . So, we should get to something that will be the same integral $dx dy$. I mean, if you have a function of x and y , then it will be the same function. But, of course, the bounds change. So, how do we exchange the order of integration? Well, the only way to do it consistently is to draw a picture. So, let's see, what does this mean? Here, it means we integrate from y equals x to y equals $2x$, x between zero and one. So, we should draw a picture.

The lower bound for y is y equals x . So, let's draw y equals x . That seems to be here. And, we'll go up to y equals $2x$, which is a line also but with bigger slope. And then, all right, so for each value of x , my origin will go from x to $2x$. Well, and I do this for all values of x that go to x equals one. So, I stop at x equals one, which is here. And then, my region is something like this. OK, so this point here, in case you are wondering, well, when x equals one, y is one. And that point here is one, two. OK, any questions about that so far?

OK, so somehow that's the first kill, when you see an integral, how to figure out what it means, how to draw the region. And then there's a converse scale which is given the region, how to set up the integral for it. So, if we want to set up instead $dx dy$, then it means we are going to actually look at the converse question which is, for a given value of y , what is the range of values of x ? OK, so if we fix y , well, where do we enter the region, and where do we leave it? So, we seem to enter on this side, and we seem to leave on that side. At least that seems to be true for the first few values of y that I choose. But, hey, if I take a larger value of y , then I will enter on the side, and I will leave on this vertical side, not on that one. So, I seem to have two different things going on. OK, the place where enter my region is always y equals $2x$, which is the same as x equals y over two.

So, x seems to always start at y over two. But, where I leave to be either x equals y , or here, x equals y . And, that depends on the value of y . So, in fact, I have to break this into two different integrals. I have to treat separately the case where y is between zero and one, and between one and two. So, what I do in that case is I just make two integrals. So, I say, both of them start at y over two. But, in the first case, we'll stop at x equals y . In the second case, we'll stop at x equals one. OK, and now, what are the values of y for each case? Well, the first case is when y is between zero and one.

The second case is when y is between one and two, which I guess this picture now is completely unreadable, but hopefully you've been following what's going on, or else you can see it in the solutions to the problem. And, so that's our final answer. OK, any questions about how to set up double integrals in xy coordinates? No? OK, who feels comfortable with this kind of problem? OK, good. I'm happy to see the vast majority. So, the bad news is we have to be able to do it not only in xy coordinates, but also in polar coordinates.

So, when you go to polar coordinates, basically all you have to remember on the side of integrand is that x becomes $r \cos \theta$. y becomes $r \sin \theta$. And, $dx dy$ becomes $r dr d\theta$. In terms of how you slice for your region, well, you will be integrating first over r . So, that means what you're doing is you're fixing the value of θ . And, for that value of θ , you ask yourself, for what range of values of r am I going to be inside my origin?

So, if my origin looks like this, then for this value of θ , r would go from zero to whatever this distance is. And of course I have to find how this distance depends on θ . And then, I will find the extreme values of θ . Now, of course, if the origin is really looking like this, then you're not going to do it in polar coordinates. But, if it's like a circle or a half circle, or things like that, then even if a problem doesn't tell you to do it in polar coordinates you might want to seriously consider it.

OK, so I'm not going to do it but problem two in the practice exam is a good example of doing something in polar coordinates. OK, so in terms of things that we do with double integrals, there's a few formulas that I'd like you to remember about applications that we've seen of double integrals. So, quantities that we can compute with double integrals include things like the area of region, its mass if it has a density, the average value of some function, for example, the average value of the x and y coordinates, which we called the center of mass or moments of inertia. So, these are just formulas to remember.

So, for example, the area of region is the double integral of just dA , or if it helps you, one dA if you want. You are integrating the function 1. You have to remember formulas for mass, for the average value of a function is the \bar{F} , in particular \bar{x} \bar{y} , which is the center of mass, and the moment of inertia. OK, so the polar moment of inertia, which is moment of inertia about the origin. OK, so that's double integral of $x^2 + y^2$, density dA , but also moments of inertia about the x and y axis, which are given by just taking one of these guys.

Don't worry about moments of inertia about an arbitrary line. I will ask you for a moment of inertia for some weird line or something like that. OK, but these you should know. Now, what if you somehow, on the spur of the moment, you forget, what's the formula for moment of inertia? Well, I mean, I prefer if you know, but if you have a complete blank in your memory, there will still be partial credit were setting up the bounds and everything else.

So, the general rule for the exam will be if you're stuck in a calculation or you're missing a little piece of the puzzle, try to do as much as you can. In particular, try to at least set up the bounds of the integral. There will be partial credit for that always. So, while we're at it about grand rules, how about evaluation? How about evaluating integrals? So, once you've set it up, you have to sometimes compute it.

First of all, check just in case the problem says set up but do not evaluate. Then, don't waste your time evaluating it. If a problem says to compute it, then you have to compute it. So, what kinds of integration techniques do you need to know? So, you need to know, you must know, well, how to integrate the usual functions like one over x or x to the n , or exponential, sine, cosine, things like that, OK, so the usual integrals.

You must know what I will call easy trigonometry. OK, I don't want to give you a complete list. And the more you ask me about which ones are on the list, the more I will add to the list. But, those that you know that you should know, you should know. Those that you think you shouldn't know, you don't have to know because I will say what I will say soon. You should know also substitution, how to set U equals something, and then see, oh, this becomes u times du , and so substitution method. What do I mean by easy trigonometrics? Well, certainly you should know how to integrate sine. You should know how to integrate cosine. You should be aware that sine squared plus cosine squared simplifies to one.

And, you should be aware of general things like that. I would like you to know, maybe, the double angles, sine $2x$ and cosine $2x$. Know what these are, and the kinds of the easy things you can do with that, also things that involve substitution setting like U equals sine T or U equals cosine T . I mean, let me, instead, give an example of hard trig that you don't need to know, and then I will answer.

OK, so, not needed on Thursday; it doesn't mean that I don't want you to know them. I would love you to know every single integral formula. But, that shouldn't be your top priority. So, you don't need to know things like hard trigonometric ones. So, let me give you an example. OK, so if I ask you to do this one, then actually I will give you maybe, you know, I will reprint the formula from the notes or something like that. OK, so that one you don't need to know. I would love if you happen to know it, but if you need it, it will be given to you.

So, these kinds of things that you cannot compute by any easy method. And, integration by parts, I believe that I successfully test-solved all the problems without doing any single integration by parts. Again, in general, it's something that I would like you to know, but it shouldn't be a top priority for this week. OK, sorry, you had a question, or? Inverse trigonometric functions: let's say the most easy ones. I would like you to know the easiest inverse trig functions, but not much. OK, OK, so be aware that these functions exist, but it's not a top priority.

I should say, the more I tell you I don't need you to know, the more your physics and other teachers might complain that, oh, these guys don't know how to integrate. So, try not to forget everything. But, yes? No, no, here I just mean for evaluating just a single variable integral. I will get to change variables and Jacobian soon, but I'm thinking of this as a different topic. What I mean by this one is if I'm asking you to integrate, I don't know, what's a good example?

Zero to one $t \, dt$ over square root of one plus t squared, then you should think of maybe substituting u equals one plus t squared, and then it becomes easier. OK, so this kind of trig, that's what I have in mind here specifically. And again, if you're stuck, in particular, if you hit this dreaded guy, and you don't actually have a formula giving you what it is, it means one of two things. One is something's wrong with your solution. The other option is something is wrong with my problem. So, either way, check quickly what you've done it if you can't find a mistake, then just move ahead to the next problem.

Which one, this one? Yeah, I mean if you can do it, if you know how to do it, which everything is fair: I mean, generally speaking, give enough of it so that you found the solution by yourself, not like, you know, it didn't somehow come to you by magic. But, yeah, if you know how to integrate this without doing the substitution, that's absolutely fine by me. Just show enough work. The general rule is show enough work that we see that you knew what you are doing.

OK, now another thing we've seen with double integrals is how to do more complicated changes of variables. So, when you want to replace x and y by some variables, u and v , given by some formulas in terms of x and y . So, you need to remember basically how to do them. So, you need to remember that the method consists of three steps. So, one is you have to find the Jacobian. And, you can choose to do either this Jacobian or the inverse one depending on what's easiest given what you're given. You don't have to worry about solving for things the other way around. Just compute one of these Jacobians.

And then, the rule is that $du \, dv$ is absolute value of the Jacobian $dx \, dy$. So, that takes care of $dx \, dy$, how to convert that into $du \, dv$. The second thing to know is that, well, you need to of course substitute any x and y 's in the integrand to convert them to u 's and v 's so that you have a valid integrand involving only u and v . And then, the last part is setting up the bounds. And you see that, probably you seen on P-sets and an example we did in the lecture that this can be complicated. But now, in real life, you do this actually to simplify the integrals. So, probably the one that will be there on Thursday, if there's a problem about that on Thursday, it will be a situation where the bounds that you get after changing variables are reasonably easy. OK, I'm not saying that it will be completely obvious necessarily, but it will be a fairly easy situation. So, the general method is you look at your region, R , and it might have various sides.

Well, on each side you ask yourself, what do I know about x and y , and how to convert that in terms of u and v ? And maybe you'll find that the equation might be just u equals zero for example, or u equals v , or something like that. And then, it's up to you to decide what you want to do. But, maybe the easiest usually is to draw a new picture in terms of u and v coordinates of what your region will look like in the new coordinates. It might be that it will actually be much easier. It should be easier looking than what you started with.

OK, so that's the general idea. There is one change of variable problem on each of the two practice exams to give you a feeling for what's realistic. The problem that's on practice exam 3B actually is on the hard side of things because the question is kind of hidden in a way. So, if you look at problem six, you might find that it's not telling you very clearly what you have to do. That's because it was meant to be the hardest problem on that test.

But, once you've reduced it to an actual change of variables problem, I expect you to be able to know how to do it. And, on practice exam 3A, there's also, I think it's problem five on the other practice exam. And, that one is actually pretty standard and straightforward. OK, time to move on, sorry. So, we've also seen about line integrals. OK, so line integrals, so the main thing to know about them, so the line integral for work, which is line integral of $F \cdot dr$, so let's say that your vector field has components, M and N . So, the line integral for work becomes in coordinates integral of $M dx$ plus $N dy$ while we've also seen line integral for flux.

So, line integral of $F \cdot n \, ds$ becomes the integral along C just to make sure that I give it to you correctly. So, remember that just, I don't want to make the mistake in front of you. So, $T \, ds$ is dx, dy . And, the normal vector, so, $T \, ds$ goes along the curve. $N ds$ goes clockwise perpendicular to the curve. So, it's going to be, well, it's going to be dy and negative dx . So, you will be integrating negative $N dx$ plus $M dy$.

OK, see, if you are blanking and don't remember the signs, then you can just draw this picture and make sure that you get it right. So, you should know a little bit about geometric interpretation and how to see easily that it's going to be zero in some cases. But, mostly you should know how to compute, set up and compute these things. So, what do we do when we are here? Well, it's year, we have both x and y together, but we want to, because it's the line integral, there should be only one variable. So, the important thing to know is we want to reduce everything to a single parameter.

OK, so the evaluation method is always by reducing to a single parameter. So, for example, maybe x and y are both functions of some variable, t , and then express everything in terms of some integral of, some quantity involving $t \, dt$. It could be that you will just express everything in terms of x or in terms of y , or in terms of some angle or something. It's up to you to choose how to parameterize things. And then, when you're there, it's a usual one variable integral with a single variable in there.

OK, so that's the general method of calculation, but we've seen a shortcut for work when we can show that the field is the gradient of potential. So, one thing to know is if the curl of F , which is x minus M_y happens to be zero, well, and now I can say, and the domain is simply connected, or if the field is defined everywhere, then F is actually a gradient field. So, that means, just to make it more concrete, that means we can find a function little f called the potential such that its derivative respect to x is M , and its derivative with respect to Y is N .

We can solve these two conditions for the same function, f , simultaneously. And, how do we find this function, little f ? OK, so that's the same as saying that the field, big F , is the gradient of little f . And, how do we find this function, little f ? Well, we've seen two methods. One of them involves computing a line integral from the origin to a point in the plane by going first along the x axis, then vertically.

The other method was to first figure out what this one tells us by integrating it with respect to x . And then, we differentiate our answer with respect to y , and we compare with that to get the complete answer. OK, so is that relevant? Well, first of all it's relevant in physics, but it's also relevant just to calculation of line integrals because we see the fundamental theorem of calculus for line integrals which says if we are integrating a gradient field and we know what the potential is.

Then, we just have to, well, the line integral is just the change in value of a potential. OK, so we take the value of a potential at the starting point, sorry, we take value potential at the endpoint minus the value at the starting point. And, that will give us the line integral, OK? So, important: this is only for work. There's no statement like that for flux, OK, so don't try to fly this in a problem about flux. I mean, usually, if you look at the practice exams, you will see it's pretty clear that there's one problem in which you are supposed to do things this way. It's kind of a dead giveaway, but it's probably not too bad.

OK, and the other thing we've seen, so I mentioned it at the beginning but let me mention it again. To compute things, Green's theorem, let's just compute, well, let us forget, sorry, find the value of a line integral along the closed curve by reducing it to double integral. So, the one for work says -- -- this, and you should remember that in there, so C is a closed curve that goes counterclockwise, and R is the region inside. So, the way you would, if you had to compute both sides separately, you would do them in extremely different ways, right? This one is a line integral. So, you use the method to explain here, namely, you express x and y in terms of a single variable. See that you're doing a circle. I want to see a θ . I don't want to see an R . R is not a variable. You are on the circle. This one is a double integral. So, if you are doing it, say, on a disk, you would have both R and θ if you're using polar coordinates. You would have both x and y .

Here, you have two variables of integration. Here, you should have only one after you parameterize the curve. And, the fact that it stays $\text{curl } F$, I mean, $\text{curl } F$ is just $N_x - M_y$ is just like any function of x and y . OK, the fact that we called it $\text{curl } F$ doesn't change how you compute it. You have first to compute the curl of F . Say you find, I don't know, xy minus x squared, well, it becomes just the usual double integral of the usual function xy minus x squared.

There's nothing special to it because it's a curl. And, the other one is the counterpart for flux. So, it says this, and remember this is mx plus ny . I mean, what's important about these statements is not only remembering, you know, if you just know this formula by heart, you are still in trouble because you need to know what actually the symbols in here mean. So, you should remember, what is this line integral, and what's the divergence of a field? So, just something to remember. And, so I guess I'll let you figure out practice problems because it's time, but I think that's basically the list of all we've seen. And, well, that should be it.

So, again, I will have office hours now, tomorrow, and otherwise, good luck on the test.