 All right, so the past few weeks, we've been looking at double integrals and the plane, line integrals in the plane, and will we are going to do now from now on basically until the end of the term, will be very similar stuff, but in space. So, we are going to learn how to do triple integrals in space, flux in space, work in space, divergence, curl, all that. So, that means, basically, if you were really on top of what we've been doing these past few weeks, then it will be just the same with one more coordinate. And, you will see there are some differences. But, conceptually, it's pretty similar. There are a few tricky things, though. Now, that also means that if there is stuff that you are not sure about in the plane, then I encourage you to review the material that we've done over the past few weeks to make sure that everything in the plane is completely clear to you because it will be much harder to understand stuff in space if things are still shaky in the plane. OK, so the plan is we're going to basically go through the same stuff, but in space. So, it shouldn't be surprising that we will start today with triple integrals. OK, so the way triple integrals work is if I give you a function of three variables, x, y, z, and I give you some region in space, so, some solid, then I can take the integral over this region over function f dV where dV stands for the volume element. OK, so what it means is we will just take every single little piece of our solid, take the value of f there, multiply by the small volume of each little piece, and sum all these things together. And, so this volume element here, well, for example, if you are doing the integral in rectangular coordinates, that will become dx dy dz or any permutation of that because, of course, we have lots of possible orders of integration to choose from. So, rather than bore you with theory and all sorts of complicated things, let's just do examples. And, you will see, basically, if you understand how to set up iterated integrals into variables, that you basically understand how to do them in three variables. You just have to be a bit more careful. And, there's one more step. OK, so let's take our first triple integral to be on the region. So, of course, there's two different things as always. There is the region of integration and there's the function we are integrating. Now, the function we are integrating, well, it will come in handy when you actually try to evaluate the integral. But, as you can see, probably, the new part is really hard to set it up. So, the function won't really matter that much for me. So, in the examples I'll do today, functions will be kind of silly. So, for example, let's say that we want to look at the region between two paraboloids, one given by z = x ^2 y ^2. The other is z = 4 - x ^2 - y ^2. And, so, I haven't given you, yet, the function to integrate. OK, this is not the function to integrate. This is what describes the region where I will integrate my function. And, let's say that I just want to find the volume of this region, which is the triple integral of just one dV. OK, similarly, remember, when we try to find the area of the region in the plane, we are just integrating one dA. Here we integrate one dV. that will give us the volume. Now, I know that you can imagine how to actually do this one as a double integral. But, the goal of the game is to set up the triple integral. It's not actually to find the volume. So, what does that look like? Well, z = x ^2 y ^2, that's one of our favorite paraboloids. That's something that looks like a parabola with its bottom at the origin that you spin about the z axis. And, z equals four minus x squared minus y squared, well, that's also a paraboloid. But, this one is pointing down, and when you take x equals y equals zero, you get z equals four. So, it starts at four, and it goes down like that. OK, so the solid that we'd like to consider is what's in between in here. So, it has a curvy top which is this downward paraboloid, a curvy bottom which is the other paraboloid. And, what about the sides? Well, do you have any idea what we get here? Yeah, it's going to be a circle because entire picture is invariant by rotation about the z axis. So, if you look at the picture just, say, in the yz plane, you get this point and that point. And, when you rotate everything around the z axis, you will just get a circle here. OK, so our goal is to find the volume of this thing, and there's lots of things I could do to simplify the calculation, or even not do it as a triple integral at all. But, I want to actually set it up as a triple integral just to show how we do that. OK, so the first thing we need to do is choose an order of integration. And, here, well, I don't know if you can see it yet, but hopefully soon that will be intuitive to you. I claim that I would like to start by integrating first over z. What's the reason for that? Well, the reason is if I give you x and y, then you can find quickly, what's the bottom and top values of z for that choice of x and y? OK, so if I have x and y given, then I can find above that: what is the bottom z and the top z corresponding to the vertical line above that point? The portion of it that's inside our solid, so somehow, there's a bottom z and a top z. And, so the top z is actually on the downward paraboloid. So, it's four minus x squared minus y squared. The bottom value of z is x squared plus y squared. OK, so if I want to start to set this up, I will write the triple integral. And then, so let's say I'm going to do it dz first, and then, say, dy dx. It doesn't really matter. So then, for a given value of x and y, I claim z goes from the bottom surface. The bottom face is z equals x squared plus y squared. The top face is four minus x squared minus y squared. OK, is that OK with everyone? Yeah? Any questions so far? Yes? Why did I start with z? That's a very good question. So, I can choose whatever order I want, but let's say I did x first . Then, to find the inner integral bounds, I would need to say, OK, I've chosen values of, see, in the inner integral, you've fixed the two other variables, and you're just going to vary that one. And, you need to find bounds for it. So, if I integrate over x first, I have to solve, answer the following question. Say I'm given values of y and z. What are the bounds for x? So, that would mean I'm slicing my solid by lines that are parallel to the x axis. And, see, it's kind of hard to find, what are the values of x at the front and at the back? I mean, it's possible, but it's easier to actually first look for z at the top and bottom. Yes? dy dx, or dx dy? No, it's completely at random. I mean, you can see x and y play symmetric roles. So, if you look at it, it's reasonably clear that z should be the easiest one to set up first for what comes next. xy or yx, it's the same. Yes? Yes, it will be easier to use cylindrical coordinates. I'll get to that just as soon as I'm done with this one. OK, so let's continue a bit with that. And, as you mentioned, actually we don't actually want to do it with xy in the end. In a few minutes, we will actually switch to cylindrical coordinates. But, for now, we don't even know what they are. OK, so I've done the inner integral by looking at, you know, if I slice by vertical lines, what is the top? What is the bottom for a given value of x and y? So, the bounds in the inner integral depend on both the middle and outer variables. Next, I need to figure out what values of x and y I will be interested in. And, the answer for that is, well, the values of x and y that I want to look at are all those that are in the shade of my region. So, in fact, to set up the middle and outer bounds, what I want to do is project my solid. So, my solid looks like this kind of thing. And, I don't really know how to call it. But, what's interesting now is I want to look at the shadow that it casts in the xy plane. OK, and, of course, that shadow will just be the disk that's directly below this disk here that's separating the two halves of the solid. And so, now I will want to integrate over, I want to look at all the xy's, x and y, in the shadow. So, now I'm left with, actually, something we've already done, namely setting up a double integral over x and y. So, if it helps, here, we don't strictly need it, but if it helps, it could be useful to actually draw a picture of this shadow in the xy plane. So, here it would just look, again, like a disk, and set it up. Now, the question is, how do we find the size of this disk, the size of the shadow? Well, basically we have to figure out where our two paraboloids intersect. There's nothing else. OK, so, one way how to find the shadow in the xy plane -- -- well, here we actually know the answer a priori, but even if we didn't, we could just say, well, our region lives wherever the bottom surface is below the top surface, OK, so we want to look at things wherever bottom value of z is less than the top value of z, I mean, less or less than or equal, that's the same thing. So, if the bottom value of z is x squared plus y squared should be less than four minus x squared minus y squared, and if you solve for that, then you will get, well, so let's move these guys over here. You'll get two x squared plus two y squared less than four. That becomes x squared plus y squared less than two. So, that means that's a disk of radius square root of two, OK? So, we kind of knew in advance it was going to be a disk, but what we've learned now is that this radius is square root of two. So, if we want to set up, if we really want to set it up using dy dx like they started, then we can do it because we know, so, for the middle integral, now, we want to fix a value of x. And, for that fixed value of x, we want to figure out the bounds for y. Well, the answer is y goes from here to here. What's here? Well, here, y is square root of two minus x squared. And, here it's negative square root of two minus x squared. So, y will go from negative square root of two minus x squared to positive square root. And then, x will go from negative root two to root two. OK, if that's not completely clear to you, then I encourage you to go over how we set up double integrals again. OK, does that make sense, kind of? Yeah? Well, so, when we set up, remember, we are setting up a double integral, dy dx here. So, when we do it dy dx, it means we slice this region of a plane by vertical line segments. So, this middle guy would be what used to be the inner integral. So, in the inner, remember, you fix the value of x, and you ask yourself, what is the range of values of y in my region? So, y goes from here to here, and what here and here are depends on the value of x. How? Well, we have to find the relation between x and y at these points. These points are on the circle of radius root two. So, if you want this circle maybe I should have written, is x squared plus y squared equals two. And, if you solve for y, given x, you get plus minus root of two minus x squared, OK? Yes? Is there a way to compute this with symmetry? Well, certainly, yeah, this solid looks sufficiently symmetric, but actually you could certainly, if you don't want to do the whole disk, you could just do quarter disks, and multiply by four. You could even just look at the lower half of the solid, and multiply them by two, so, total by eight. So, yeah, certainly there's lots of ways to make it slightly easier by using symmetry. Now, the most spectacular way to use symmetry here, of course, is to use that we have this rotation symmetry and switch, actually, not do this guy in xy coordinates but instead in polar coordinates. So -- So, the smarter thing to do would be to use polar coordinates instead of x and y. Of course, we want to keep z. I mean, we are very happy with z the way it is. But, we'll just change x and y to R cos theta, R sine theta, OK, because, well, let's see actually how we would evaluate this guy. So, well actually, let's not. It's kind of boring. So, let me just point out one small thing here, sorry, before I do that. So, if you start computing the inner integral, OK, so let me not do that yet, sorry, so if you try to compute the inner integral, you'll be integrating from x squared plus y squared to four minus x squared minus y squared dz. Well, that will integrate to z between these two bounds. So, you will get four minus two x squared minus two y squared. Now, when you put that into the remaining ones, you'll get something that's probably not very pleasant of four minus two x squared minus two y squared dy dx. And here, you see that to evaluate this, you would switch to polar coordinates. Oh, by the way, so if your initial instincts had been to, given that you just want the volume, you could also have found the volume just by doing a double integral of the height between the top and bottom. Well, you would just have gotten this, right, because this is the height between top and bottom. So, it's all the same. It doesn't really matter. But with this, of course, we will be able to integrate all sorts of functions, not just one over the solid. So, we will be able to do much more than just volumes. OK, so let's see, how do we do it with polar coordinates instead? Well, so -- Well, that would become, so let's see. So, I want to keep dz. But then, dx dy or dy dx would become r dr d theta. And, if I try to set up the bounds, well, I probably shouldn't keep this x squared plus y squared around. But, x squared plus y squared is easy in terms of r and theta. That's just r squared. OK, I mean, in general I could have something that depends also on theta. That's perfectly legitimate. But here, it simplifies, and this guy up here, four minus x squared minus y squared becomes four minus r squared. And now, the integral that we have to do over r and theta, well, we look again at the shadow. The shadow is still a disk of radius root two. That hasn't changed. And now, we know how to set up this integral in polar coordinates. r goes from zero to root two, and theta goes from zero to two pi. OK, and now it becomes actually easier to evaluate. OK, so now we have actually a name for this because we're doing it in space. So, these are called, actually, cylindrical coordinates. So, in fact, you already knew about cylindrical coordinates even if you did not know the name. OK, so the idea of cylindrical coordinates is that instead of x, y, and z, to locate a point in space, you will use three coordinates. One of them is basically how high it is above the xy plane. So, that will be z. And then, you will use polar coordinates for the projection of your point on the xy plane. So, r will be the distance from the z axis. And theta will be the angle from the x axis counterclockwise. So, the one thing to be careful about is because of the usual convention, that we make the x axis point toward us. Theta equals zero is no longer to the right. Now, theta equals zero is to the front, and the angel is measured from the front counterclockwise. OK, so, and of course, if you want to know how to convert between x, y, z and r theta z, well, the formulas are just the same as in usual polar coordinates. R cos theta, r sine theta, and z remain z. OK, so why are these called cylindrical coordinates, by the way? Well, let's say that I gave you the equation r equals a, where a is some constant. Say r equals one, for example. So, r equals one in 2D, that used to be just a circle of radius one. Now, in space, a single equation actually defines a surface, not just a curve anymore. And, the set of points where r is a, well, that's all the points that are distance a from the z axis. So, in fact, what you get this way is a cylinder of radius a centered on the z axis. OK, so that's why they are called cylindrical coordinates. By the way, so now, similarly, if you look at the equation theta equals some given value, well, so that used to be just a ray from the origin. Now, that becomes a vertical half plane. For example, if I set the value of theta and let r and z vary, well, r is always positive, but basically that means I am taking a vertical plane that comes out in this direction. OK, any questions about cylindrical coordinates? Yes? Yeah, so I'm saying when you fix theta, you get only a half plane, not a full plane. I mean, it goes all the way up and down, but it doesn't go back to the other side of the z axis. Why? That's because r is always positive by convention. So, for example, here, we say theta is zero. At the back, we say theta is pi. We don't say theta is zero and r is negative. We say r is positive and theta is pi. It's a convention, largely. But, sticking with this convention really will help you to set up the integrals properly. I mean, otherwise there is just too much risk for mistakes. Yes? Well, so the question is if I were to use symmetry to do this one, would I multiply by four or by two? Well, it depends on how much symmetry you are using. So, I mean, it's your choice. You can multiply by two, by four, by eight depending on how much you cut it. So, it depends on what symmetry you use, if you use symmetry between top and bottom you'd say, well, the volume is twice the lower half. If you use the left and right half, you would say it's twice each half. If you cut it into four pieces, and so on. So, and again, you don't have to use the symmetry. If you don't think of using polar coordinates, then it can save you from doing, you know, you can just start at zero here and here, and simplify things a tiny bit. But, OK, yes? So, to define a vertical full plane, well, first of all it depends on whether it passes through the z axis or not. If it doesn't, then you'd have to remember how you do in polar coordinates. I mean, basically the answer is, if you have a vertical plane, so, it doesn't depend on z. The equation does not involve z. It only involves r and theta. And, how it involves r and theta is exactly the same as when you do a line in polar coordinates in the plane. So, if it's a line passing through the origin, you say, well, theta is either some value or the other one. If it's a line that doesn't passes to the origin, but it's more tricky. But hopefully you've seen how to do that. OK, let's move on a bit. So, one thing to know, I mean, basically, the important thing to remember is that the volume element in cylindrical coordinates, well, dx dy dz becomes r dr d theta dz. And, that shouldn't be surprising because that's just dx dy becomes r dr d theta. And, dz remains dz. I mean, so, the way to think about it, if you want, is that if you take a little piece of solid in space, so it has some height, delta z, and it has a base which has some area delta A, then the small volume, delta v, is equal to the area of a base times the height. So, now, when you make the things infinitely small, you will get dV is dA times dz, and you can use whichever formula you want for area in the xy plane. OK, now in practice, you choose which order you integrate in. As you have probably seen, a favorite of mine is z first because very often you'll know what the top and bottom of your solid look like, and then you will reduce to just something in the xy plane. But, there might be situations where it's actually easier to start first with dx dy or r dr d theta, and then save dz for last. I mean, if you seen how to, in single variable calculus, the disk and shell methods for finding volumes, that's exactly the dilemma of shells versus disks. One of them is you do z first. The other is you do z last. OK, so what are things we can do now with triple integrals? Well, we can find the volume of solids by just integrating dV. And, we've seen that. We can find the mass of a solid. OK, so if we have a density, delta, which, remember, delta is basically the mass divided by the volume. OK, so the small mass element, maybe I should have written that as dm, the mass element, is density times dV. So now, this is the real physical density. If you are given a material, usually, the density will be in grams per cubic meter or cubic inch, or whatever. I mean, there is tons of different units. But, so then, the mass of your solid will be just the triple integral of density, dV because you just sum the mass of each little piece. And, of course, if the density is one, then it just becomes the volume. OK, now, it shouldn't be surprising to you that we can also do classics that we had seen in the plane such as the average value of a function, the center of mass, and moment of inertia. OK, so the average value of the function f of x, y, z in the region, r, that would be f bar, would be one over the volume of the region times the triple integral of f dV. Or, if we have a density, and we want to take a weighted average -- Then we take one over the mass where the mass is the triple integral of the density times the triple integral of f density dV. So, as particular cases, there is, again, the notion of center of mass of the solid. So, that's the point that somehow right in the middle of the solid. That's the point mass by which there is a point at which you should put point mass so that it would be equivalent from the point of view of dealing with forces and translation effects, of course, not for rotation. But, so the center of mass of a solid is just given by taking the average values of x, y, and z. OK, so there is a special case where, so, x bar is one over the mass times triple integral of x density dV. And, same thing with y and z. And, of course, very often, you can use symmetry to not have to compute all three of them. For example, if you look at this solid that we had, well, I guess I've erased it now. But, if you remember what it looked, well, it was pretty obvious that the center of mass would be in the z axis. So, no need to waste time considering x bar and y bar. And, in fact, you can also find z bar by symmetry between the top and bottom, and let you figure that out. Of course, symmetry only works, I should say, symmetry only works if the density is also symmetric. If I had taken my guy to be heavier at the front than at the back, then it would no longer be true that x bar would be zero. OK, next on the list is moment of inertia. Actually, in a way, moment of inertia in 3D is easier conceptually than in 2D. So, why is that? Well, because now the various flavors that we had come together in a nice way. So, the moment of inertia of an axis, sorry, with respect to an axis would be, again, given by the triple integral of the distance to the axis squared times density, times dV. And, in particular, we have our solid. And, we might skewer it using any of the coordinate axes and then try to rotate it about one of the axes. So, we have three different possibilities, of course, the x, y, or z axis. And, so now, rotating about the z axis actually corresponds to when we were just doing things for flat objects in the xy plane. That corresponded to rotating about the origin. So, secretly, we were saying we were rotating about the point. But actually, it was just rotating about the z axis. Just I didn't want to introduce the z coordinate that we didn't actually need at the time. So -- [APPLAUSE] OK, so moment of inertia about the z axis, so, what's the distance to the z axis? Well, we've said that's exactly r. That's the cylindrical coordinate, r. So, the square of a distance is just r squared. Now, if you didn't want to do it in cylindrical coordinates then, of course, r squared is just x squared plus y squared. Square of distance from the z axis is just x squared plus y squared. Similarly, now, if you want the distance from the x axis, well, that will be y squared plus z squared. OK, try to convince yourselves of the picture, or else just argue by symmetry: you know, if you change the positions of the axis. So, moment of inertia about the x axis is the double integral of y squared plus z squared delta dV. And moment of inertia about the y axis is the same thing, but now with x squared plus z squared. And so, now, if you try to apply these things for flat solids that are in the xy plane, so where there's no z to look at, well, you see these formulas become the old formulas that we had. But now, they all fit together in a more symmetric way. OK, any questions about that? No? OK, so these are just formulas to remember. So, OK, let's do an example. Was there a question that I missed? No? OK, so let's find the moment of inertia about the z axis of a solid cone -- -- between z equals a times r and z equals b. So, just to convince you that it's a cone, so, z equals a times r means the height is proportional to the distance from the z axis. So, let's look at what we get if we just do it in the plane of a blackboard. So, if I go to the right here, r is just the distance from the x axis. The height should be proportional with proportionality factor A. So, that means I take a line with slope A. If I'm on the left, well, it's the same story except distance to the z axis is still positive. So, I get the symmetric thing. And, in fact, it doesn't matter which vertical plane I do it in. This is the same if I rotate about. See, there's no theta in here. So, it's the same in all directions. So, I claim it's a cone where the slope of the rays is A. OK, and z equals b. Well, that just means we stop in our horizontal plane at height b. OK, so that's solid cone really just looks like this. That's our solid. OK, so it has a flat top, that circular top, and then the point is at v. The tip of it is at the origin. So, let's try to compute its moment of inertia about the z axis. So, that means maybe this is like the top that you are going to spin. And, it tells you how hard it is to actually spin that top. Actually, that's also useful if you're going to do mechanical engineering because if you are trying to design gears, and things like that that will rotate, you might want to know exactly how much effort you'll have to put to actually get them to spin, and whether you're actually going to have a strong enough engine, or whatever, to do it. OK, so what's the moment of inertia of this guy? Well, that's the triple integral of, well, we have to choose x squared plus y squared or r squared. Let's see, I think I want to use cylindrical coordinates to do that, given the shape. So, we use r squared. I might have a density that let's say the density is one. So, I don't have density. I still have dV. Now, it will be my choice to choose between doing the dz first or doing r dr d theta first. Just to show you how it goes the other way around, let me do it r dr d theta dz this time. Then you can decide on a case-by-case basis which one you like best. OK, so if we do it in this direction, it means that in the inner and middle integrals, we've fixed a value of z. And, for that particular value of z, we'll be actually slicing our solid by a horizontal plane, and looking at what we get, OK? So, what does that look like? Well, I fixed a value of z, and I slice my solid by a horizontal plane. Well, I'm going to get a circle certainly. What's the radius, well, a disk actually, what's the radius of the disk? Yeah, the radius of the disk should be z over a because the equation of that cone, we said it's z equals ar. So, if you flip it around, so, maybe I should switch to another blackboard. So, the equation of a cone is z equals ar, or equivalently r equals z over a. So, for a given value of z, I will get, this guy will be a disk of radius z over a. OK, so, moment of inertia is going to be, well, we said r squared, r dr d theta dz. Now, so, to set up the inner and middle integrals, I just set up a double integral over this disk of radius z over a. So, it's easy. r goes from zero to z over a. Theta goes from zero to 2pi. OK, and then, well, if I set up the bounds for z, now it's my outer variable. So, the question I have to ask is what is the first slice? What is the last slice? So, the bottommost value of z would be zero, and the topmost would be b. And so, that's it I get. So, exercise, it's not very hard. Try to set it up the other way around with dz first and then r dr d theta. It's pretty much the same level of difficulty. I'm sure you can do both of them. So, and also, if you want to practice calculations, you should end up getting pi b to the five over 10a to the four if I got it right. OK, let me finish with one more example. I'm trying to give you plenty of practice because in case you haven't noticed, Monday is a holiday. So, you don't have recitation on Monday, which is good. But it means that there will be lots of stuff to cover on Wednesday. So -- Thank you. OK, so third example, let's say that I want to just set up a triple integral for the region where z is bigger than one minus y inside the unit ball centered at the origin. So, the unit ball is just, you know, well, stay inside of the unit sphere. So, its equation, if you want, would be x squared plus y squared plus z squared less than one. OK, so that's one thing you should remember. The equation of a sphere centered at the origin is x squared plus y squared plus z squared equals radius squared. And now, we are going to take this plane, z equals one minus y. So, if you think about it, it's parallel to the x axis because there's no x in its coordinate in its equation. At the origin, the height is one. So, it starts right here at one. And, it slopes down with y with slope one. OK, so it's a plane that comes straight out here, and it intersects the sphere, so here and here, but also at other points in between. Any idea what kind of shape this is? Well, it's an ellipse, but it's even more than that. It's also a circle. If you slice a sphere by a plane, you always get a circle. But, of course, it's a slanted circle. So, if you look at it in the xy plane, if you project it to the xy plane, that you will get an ellipse. OK, so we want to look at this guy in here. So, how do we do that? Well, so maybe I should actually draw quickly a picture. So, in the yz plane, it looks just like this, OK? But, if I look at it from above in the xy plane, then its shadow, well, see, it will sit entirely where y is positive. So, it sits entirely above here, and it goes through here and here. And, in fact, when you project that slanted circle, now you will get an ellipse. And, well, I don't really know how to draw it well, but it should be something like this. OK, so now if you want to try to set up that double integral, sorry, the triple integral, well, so let's say we do it in rectangular coordinates because we are really evil. [LAUGHTER] So then, the bottom surface, OK, so we do it with z first. So, the bottom surface is the slanted plane. So, the bottom value would be z equals one minus y. The top value is on the sphere. So, the sphere corresponds to z equals square root of one minus x squared minus y squared. So, you'd go from the plane to the sphere. And then, to find the bounds for x and y, you have to figure out what exactly, what the heck is this region here? So, what is this region? Well, we have to figure out, for what values of x and y the plane is below the ellipse. So, the condition is that, sorry, the plane is below the sphere. OK, so, that's when the plane is below the sphere. That means one minus y is less than square root of one minus x squared minus y squared. So, you have to somehow manipulate this to extract something simpler. Well, probably the only way to do it is to square both sides, one minus y squared should be less than one minus x squared minus y squared. And, if you work hard enough, you'll find quite an ugly equation. But, you can figure out what are, then, the bounds for x given y, and then set up the integral? So, just to give you a hint, the bounds on y will be zero to one. The bounds on x, well, I'm not sure you want to see them, but in case you do, it will be from negative square root of 2y minus 2y squared to square root of 2y minus 2y squared. So, exercise, figure out how I got these by starting from that. Now, of course, if we just wanted the volume of this guy, we wouldn't do it this way. We do symmetry, and actually we'd rotate the thing so that our spherical cap was actually centered on the z axis because that would be a much easier way to set it up. But, depending on what function we are integrating, we can't always do that.