18.02 Practice Exam 3 A – Solutions

1. a) The area of the triangle is 2, so $\bar{y} = \dfrac{1}{2}\displaystyle\int_0^1 \int_{2y-2}^{2-2y} y\,dxdy$.

b) By symmetry $\bar{x} = 0$.

2. $\delta = |x| = r|\cos\theta|$. $I_0 = \displaystyle\iint_D r^2\,\delta\,rdrd\theta =$

$$\int_0^{2\pi}\int_0^1 r^2|r\cos\theta|rdrd\theta = 4\int_0^{\pi/2}\int_0^1 r^4\cos\theta drd\theta = 4\int_0^{\pi/2}\frac{1}{5}\cos\theta d\theta = \frac{4}{5}$$

3. a) $N_x = 6x^2 + by^2$, $M_y = ax^2 + 3y^2$. $N_x = M_y$ provided $a = 6$ and $b = 3$.

b) $f_x = 6x^2y + y^3 + 1 \implies f = 2x^3y + xy^3 + x + c(y)$. Therefore, $f_y = 2x^3 + 3xy^2 + c'(y)$. Setting this equal to N, we have $2x^3 + 3xy^2 + c'(y) = 2x^3 + 3xy^2 + 2$ so $c'(y) = 2$ and $c = 2y$. So

$$f = 2x^3y + xy^3 + x + 2y \quad (+\text{constant}).$$

c) C starts at $(1,0)$ and ends at $(-e^{\pi}, 0)$, so $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(-e^{\pi},0) - f(1,0) = -e^{-\pi} - 1$.

4. $\displaystyle\int_C yx^3dx + y^2dy = \int_0^1 x^2x^3dx + (x^2)^2(2xdx) = \int_0^1 3x^5dx = 1/2$.

5. a) $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x/y & -x^2/y^2 \\ y & x \end{vmatrix} = 3x^2/y$. Therefore,

$$dudv = (3x^2/y)dxdy = 3u\ dxdy \qquad \Longrightarrow \qquad dxdy = \frac{1}{3u}dudv.$$

b) $\displaystyle\int_2^4\int_1^5 \frac{1}{3u}dudv = \int_2^4 \frac{1}{3}\ln 5\ dv = \frac{2}{3}\ln 5$.

6. a) $\displaystyle\oint_C M dx = \iint_R -M_y\,dA$.

b) We want M such that $-M_y = (x+y)^2$. Use $M = -\dfrac{1}{3}(x+y)^3$.

7. a) $\operatorname{div}\vec{F} = 2y$, so $\displaystyle\iint_R 2y\,dA = \int_0^1\int_0^{x^3} 2y\,dydx = \int_0^1 x^6dx = \frac{1}{7}$.

b) For the flux through C_1, $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$ implies $\vec{F}\cdot\hat{\mathbf{n}} = -(1+y^2) = -1$ where $y = 0$. The length of C_1 is 1, so the total flux through C_1 is -1.

The flux through C_2 is zero because $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ and $\vec{F} \perp \hat{\mathbf{i}}$.

c) $\displaystyle\int_{C_3}\vec{F}\cdot\hat{\mathbf{n}}ds = \iint_R \operatorname{div}\vec{F}dA - \int_{C_1}\vec{F}\cdot\hat{\mathbf{n}}ds - \int_{C_2}\vec{F}\cdot\hat{\mathbf{n}}ds = \frac{1}{7} - (-1) - 0 = \frac{8}{7}$.