Using the Ratio Test

The ratio test for convergence is another way to tell whether a sum of the form $\sum_{n=n_0}^{\infty} a_n$, with $a_n > 0$ for all n, converges or diverges. To perform the ratio test we find the ratio $\dfrac{a_{n+1}}{a_n}$ and let:

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}.$$

The test has three possible outcomes:

$L < 1 \quad \Rightarrow \quad$ The series converges.

$L > 1 \quad \Rightarrow \quad$ The series diverges.

$L = 1 \qquad\quad$ No conclusion; the series may converge or diverge.

Apply the ratio test to each of the following series. Note that not all series satisfy the conditions needed to apply this form of the ratio test.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n}}$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

Solution

a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt[3]{n}}$

This series meets the criteria for applying the ratio test; here $a_n = \dfrac{1}{\sqrt[3]{n}} > 0$.

To apply the test we start by computing the ratio of two terms.

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(1/\sqrt[3]{n+1})}{(1/\sqrt[3]{n})} \\
&= \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} \\
&= \sqrt[3]{\frac{n}{n+1}}
\end{aligned}
$$

Next we find the limit of this ratio as n approaches infinity. When we have trouble determining the limit of a ratio of polynomials, we use polynomial

long division to convert the ratio into an "improper fraction".

$$
\begin{aligned}
L &= \lim_{n\to\infty} \sqrt[3]{\frac{n}{n+1}} \\
&= \lim_{n\to\infty} \sqrt[3]{1 - \frac{1}{n+1}} \\
&= \sqrt[3]{1} \\
L &= 1.
\end{aligned}
$$

The ratio test is inconclusive; we still do not know whether the series converges or diverges. A different test, such as the integral test for convergence, might prove more effective.

b) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

This problem provides some practice with factorials. We've seen that repeatedly differentiating a polynomial yields factorial-like integer products; this is one of many reasons why factorials often appear in series calculations. Initially, we will rewrite every factorial as a product:

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

After some practice we'll no longer need to expand them in this way.

This series satisfies the conditions needed to apply the ratio test, with $a_n = \dfrac{(n!)^2}{(2n)!} > 0$. The ratio of terms is:

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\left[((n+1)!)^2/(2(n+1))!\right]}{(n!)^2/(2n)!} \\
&= \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}
\end{aligned}
$$

We might be tempted to cancel the 2's, but the order of operations forbids it — we must first expand the factorial expressions. Note that:

$$(n+1)! = 1 \cdot 2 \cdot 3 \cdots n \cdot (n+1) = n! \cdot (n+1).$$

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n!(n+1))^2}{1 \cdot 2 \cdots 2n \cdot (2n+1) \cdot (2n+2)} \cdot \frac{1 \cdot 2 \cdots (2n-1) \cdot 2n}{(n!)^2} \\
&= \frac{(n!)^2 \cdot (n+1)^2}{(2n+1) \cdot (2n+2)} \cdot \frac{1}{(n!)^2} \\
&= \frac{(n+1)^2}{(2n+1) \cdot (2n+2)} \\
\frac{a_{n+1}}{a_n} &= \frac{n+1}{2(2n+1)}
\end{aligned}
$$

Now we take a limit to get the result of the test:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{n+1}{2(2n+1)} \\
&= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{4(2n+1)} \right) \\
L &= \frac{1}{4}
\end{aligned}
$$

Because $L < 0$ we know that this series must converge. This tells us that the value $\dfrac{1 \cdot 2 \cdots n}{(n+1)(n+2) \cdots 2n}$ of each term is very small when n is large.

c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

This series does *not* satisfy the conditions for applying the ratio test; the terms in the series are not all positive.

It is true that $\left| \displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!} \right| < \displaystyle\sum_{n=0}^{\infty} \left| \frac{(-3)^n}{n!} \right|$. (How would you prove this?)
We apply the ratio test to this second series to determine whether the first converges.

So, we have $a_n = \left| \dfrac{(-3)^n}{n!} \right| > 0$ and can apply the ratio test.

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{|(-3)^{n+1}/(n+1)!|}{|(-3)^n/n!|} \\
&= \left| \frac{(-3)^{n+1} \cdot n!}{(n+1)! \cdot (-3)^n} \right| \\
&= \left| \frac{-3}{n+1} \right| \\
\frac{a_{n+1}}{a_n} &= \frac{3}{n+1}
\end{aligned}
$$

$$
L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{n+1} = 0
$$

By the ratio test, the sum of the absolute values of the terms in the series converges, so the series itself must converge. This tells us that the value of $n!$ grows much faster than the value of 3^n.

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