Surface Area of a Sphere

In this example we will complete the calculation of the area of a surface of rotation. If we're going to go to the effort to complete the integral, the answer should be a nice one; one we can remember. It turns out that calculating the surface area of a sphere gives us just such an answer.

We'll think of our sphere as a surface of revolution formed by revolving a half circle of radius a about the x-axis. We'll be integrating with respect to x, and we'll let the bounds on our integral be x_1 and x_2 with $-a \leq x_1 \leq x_2 \leq a$ as sketched in Figure 1.

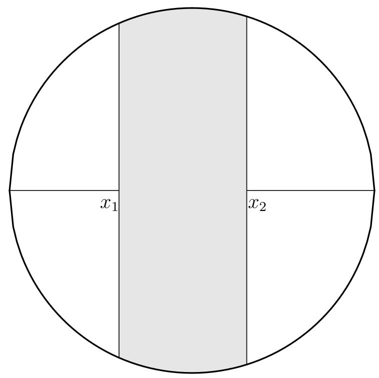

Figure 1: Part of the surface of a sphere.

Remember that in an earlier example we computed the length of an infinitesimal segment of a circular arc of radius 1:

$$ds = \sqrt{\frac{1}{1-x^2}}\,dx$$

In this example we let the radius equal a so that we can see how the surface area depends on the radius. Hence:

$$\begin{aligned}
y &= \sqrt{a^2 - x^2} \\
y' &= \frac{-x}{\sqrt{a^2 - x^2}} \\
ds &= \sqrt{1 + \frac{x^2}{a^2 - x^2}}\,dx \\
&= \sqrt{\frac{a^2 - x^2 + x^2}{a^2 - x^2}}\,dx \\
&= \sqrt{\frac{a^2}{a^2 - x^2}}\,dx.
\end{aligned}$$

The formula for the surface area indicated in Figure 1 is:

$$\text{Area} = \int_{x_1}^{x_2} 2\pi y\,ds$$

$$
\begin{aligned}
&= \int_{x_1}^{x_2} 2\pi \overbrace{\sqrt{a^2-x^2}}^{y} \overbrace{\sqrt{\frac{a^2}{a^2-x^2}}\,dx}^{ds} \\
&= \int_{x_1}^{x_2} 2\pi \sqrt{a^2-x^2}\frac{a}{\sqrt{a^2-x^2}}\,dx \\
&= \int_{x_1}^{x_2} 2\pi a\,dx \\
&= 2\pi a(x_2-x_1).
\end{aligned}
$$

Special Cases

When possible, we should test our results by plugging in values to see if our answer is reasonable. Here, if we set $x_1 = 0$ and $x_2 = a$ we should get the surface area of a hemisphere of radius a:

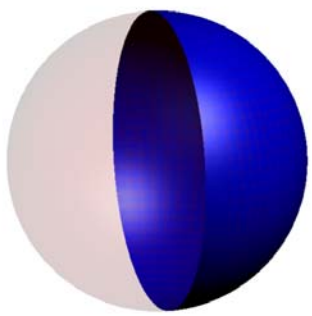

Figure 2: Right hemisphere.

$$
\begin{aligned}
2\pi a(x_2-x_1) &= 2\pi a(a-0) \\
&= 2\pi a^2
\end{aligned}
$$

We get the surface area of the whole sphere by letting $x_1 = -a$ and $x_2 = a$:

$$
\begin{aligned}
2\pi a(x_2-x_1) &= 2\pi a(a-(-a)) \\
&= 4\pi a^2
\end{aligned}
$$

Question: Would it be possible to rotate around the y-axis?

Answer: Yes. If we rotate around the y axis and integrate with respect to x (calculating the surface area of a vertical slice, as we did here) we'd be adding up little strips of area. If we integrate with respect to y and find the surface area between two vertical positions y_1 and y_2 we'd get exactly the same calculation.

Question: Can you compute surface area using shells?

Answer: The short answer is “not quite”. We use the word shell to describe something which has a thickness dx. Shells have volume, integrals which involve shells compute volumes, not surface areas.

To compute surface area you need to sum up the areas of small regions of your surface, but those small regions can have any shape whose area you can measure.