

JOEL LEWIS: Hi, welcome back to recitation. In lecture, Professor Miller did a bunch of examples of integrals involving trigonometric functions. So I thought I would give you a couple more, well I guess three more examples, some of them are a little different flavor than the ones he did, but some nice examples of some integrals you can compute now. One thing you might need for them is you're going to have to remember some of your trig identities. Just like you had to remember some of them last time. So in particular, one identity that you might need today that you didn't need in lecture, was the angle sum identity for cosine. So let me just remind you what that is, it says the cosine of a plus b is equal to cosine a cosine b minus sine a sine b. So you're going to need that formula to compute one of these three integrals.

So why don't you pause the video, take some time to work these out. Come back you can check your answers against my work.

Hopefully you had some fun and some luck working on these integrals. Let's have a go at them.

The first integral is sine cubed x secant squared x. So I don't think that Professor Miller did any integrals that involved the secant function, but one thing to remember is that the secant function is very closely related to the cosine function. It's just cosine to the minus 1. Sorry, by that I mean 1 over cosine, not the arccosine. So, in order to--

so we can view this first integral here as an integral of a power of sine times a power of cosine, just like the ones you had in lecture. So we can write this as sine cubed of x times-- I'm going to write cosine x to the minus 2 dx. And here we see this is a nice example where the sine occurs with an odd power. So when the sine occurs with an odd power, or when one of them at least occurs with an odd power, life is relatively simple. And what we do is just what we did in lecture, which is we break it up so that we just have one instance of the one that occurs to an odd power and then a bunch of powers of the other one.

So in this case, we have sine to an odd power. So we want to pull out all the even, you know, all the extra multiples, so in this case we have sine times sine squared. So we take all those sine squareds and we convert them into 1 minus cosine squareds. This is going to be equal to the integral of sine x times, well, times sine squared x, which is 1 minus cosine squared x, times cosine x to the minus second power dx. And now you take all these cosines and you multiply them together and you see what you've got.

So in this case that's equal to the integral-- OK, so 1 times cosine to the minus 2 is just cosine to the minus 2. So this first term is cosine x to the minus 2 sine x and then minus-- OK, we have cosine squared times cosine to the minus 2, so that one's even simpler, that's just minus sine x dx. So at this point, if you like, you could do a substitution of a trigonometric function. You could set u equal to cosine x. All right? What's going to happen if you set u equal to cosine x, well du is going to be minus sine x dx.

Each of these have exactly one appearance of sine x in them, so this would become-- so if we set u equal cosine x. So, as I said du equals minus sine x dx. Because each of these have a sine x dx in them, that's going to get clumped into the du. And the cosines will just get replaced with u everywhere.

So I'm going to take this over here so I have a little more space. So this becomes the integral of u to the minus 2, I guess it's minus u to the minus 2 plus 1 du. That minus sine x is just exactly equal to du. OK. And now this is easy to finish from here. This is equal to, well OK, so minus 2, so we have to go up one to minus 1, so that's just u to the minus 1, plus 1 gives us a u, plus a constant, and then of course we have to substitute back in. So this is u is cosine x, so u to the minus 1 is secant x plus cosine x plus a constant. So that's the first one.

For the second one, sine x cosine of 2x. We actually have a slightly more complicated situation here. Because the cosine 2x is no longer, it's not just a trigonometric function of x alone, it's a trigonometric function of 2x. So in order to do the sort of the things we did in lecture, what you're going to have to do is, you're going to have to expand this out in terms of just sine x and cosine x, rather than cosine of 2x.

So in order to do that you just need to remember your double angle formula. Now if you don't remember your double angle formulas, one way you can remember them is remember the angle sum formula and apply it with a and b both equal to x. If you like. But, or you could just remember your double angle formulas. So let's go over here.

And so in this case, we have the integral of sine x times, well cosine 2x-- so there are actually, right, there are several different ways you can write cosine 2x. So one way you can write it is cosine squared x minus sine squared x, but we want everything-- again we have an odd power of sine, so it would be nice if everything else could be written in terms of just cosine. So an equivalent form for cosine 2x is to write, is that it's equal to 2 cosine squared x minus 1 dx. And here, this is very, very similar to the last question. Again, you can make the substitution u

equals cosine, if you like.

And so I'm not going to do out the whole substitution for you, but this is going to work out to $2 \cos^2 x \sin x$, the integral of that is $-\frac{2}{3} \cos^3 x$ and then the integral of $-\sin x$ is $\cos x$ plus a constant of integration. So that's the second one.

The third one now is of a similar flavor, except you have here, you have $\sin 2x$ and $\cos 3x$. So again, when you want to integrate something where you have-- so here we have both trigonometric functions occur in multiples. Now, if it was the same multiple, if it were $2x$ and $2x$, say, that would be fine. You know, you could just make a substitution like u equals $2x$ or something and then proceed as usual. But because they're different multiples, we need, when we do these trig integrals, we need to apply the methods that we used in lecture. You want the arguments to agree. So you want it all to be sine of something and cosine of the same thing.

So here the thing to do, I think, is to try and put everything in terms of just $\sin x$ and $\cos x$. So $\sin 2x$ is one you should remember, double angle formula. $\cos 3x$, well in order to figure out what $\cos 3x$ is, you could just know it, maybe you learned it once upon a time. The other thing you can do for $\cos 3x$, is you can use this angle sum formula. So let me just work that out for you quickly.

So $\cos 3x$, well how do you use the angle sum formula? You can write $3x$ as $2x$ plus x . So this is equal to $\cos(2x + x)$. And now this is a sum, so you can use the cosine sum formula, so you get this is equal to $\cos 2x \cos x - \sin 2x \sin x$. OK and now you can use the double angle formulas here and here. so in the end, if you do this all out, what you get is you're going to get $4 \cos^3 x - 3 \cos x$.

You might say, oh what happened to the $\sin x$'s? We're going to get a $\sin^2 x$ and I want to write everything in terms of cosine. So I've replaced the-- I've done an extra step here that I'm not showing, where I replaced the $\sin^2 x$ with a $1 - \cos^2 x$. So you make this substitution, and you also make the substitution for $\sin x$ -- sorry, for $\sin 2x$ that you have. OK, $\sin 2x$ is equal to-- you know, double angle formula-- $2 \sin x \cos x$. And so when you multiply $\sin 2x$ by $\cos 3x$ you again have an expression that's got powers of cosine with a single power of sine. So doing these simplifications, you again get this nice form. One of them's odd, the other one-- one of the trig functions, sine or cosine appears to just the power of 1, so, OK, so you can do this nice simple substitution. You don't need to do any of the double angle, or half angle complication.

I'm not going to finish this one out for you, but I did write down the answer so you can check your answer. When you-- the third integral that we had, which is $\int \sin 2x \cos 3x \, dx$, when you work it all out you should get, let's see what I got, OK, you should get $-\frac{8}{5} \cos^5 x + 2 \cos^3 x + C$ plus a constant of integration. So if you work out all the details, this is what you should get. I'll stop there.