Area of a Smile

The region between the curves $y = \frac{1}{2}x^2 - \frac{1}{2}$ and $y = x^4 - 1$ is smile shaped. Find the area of that region.

Solution

We begin by drawing a sketch of the region whose area we are trying to find. The "lower lip" of the smile is described by $y = x^4 - 1$ and the upper lip corresponds to $y = \frac{1}{2}x^2 - \frac{1}{2}$. The smile extends from $x = -1$ to $x = 1$, and from $y = -1$ to $y = 0$. (The crossing points of the two curves are at $(-1, 0)$ and $(1, 0)$.)

Our next step is to decide whether to integrate with respect to x or with respect to y. We already have both equations written in the form $y = f(x)$, and vertical slices of the smile are much simpler than horizontal ones (which come in two parts), so it seems wisest to integrate with respect to x.

Our sketch shows us that the limits of integration are -1 and 1, that the equation of the top curve is $y = \frac{1}{2}x^2 - \frac{1}{2}$, and that the equation of the bottom curve is $y = x^4 - 1$. The area is then:

$$
\begin{aligned}
\int_{-1}^{1} \left(\frac{1}{2}x^2 - \frac{1}{2} \right) - (x^4 - 1)\, dx &= \int_{-1}^{1} -x^4 + \frac{1}{2}x^2 + \frac{1}{2}\, dx \\
&= \left[-\frac{1}{5}x^5 + \frac{1}{6}x^3 + \frac{1}{2}x \right]_{-1}^{1} \\
&= \left[\left(-\frac{1}{5} + \frac{1}{6} + \frac{1}{2} \right) - \left(\frac{1}{5} - \frac{1}{6} - \frac{1}{2} \right) \right] \\
&= \frac{14}{30} + \frac{14}{30} \\
&= \frac{14}{15} \approx 1.
\end{aligned}
$$

We check our work by comparing this answer to the area shown in our graph.

An alternate way to solve the problem is to note that the left side of the smile is identical to the right side, and to find the area of only one side of the smile then double it. To do the problem this way, evaluate:

$$
2 \int_{0}^{1} \left(\frac{1}{2}x^2 - \frac{1}{2} \right) - (x^4 - 1)\, dx
$$

to get the same final answer.

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