

Hi. Welcome back to recitation. You've been learning about infinite series of constants. Adding up, you know, series of numbers. And one thing that we've talked about are divergent series and convergent series, and rates of divergence a little bit.

So suppose you wanted to run an experiment. So suppose you started with the series $\log n$ over n , and you started summing-- I picked n equals 3. It doesn't actually matter that much-- started summing at n equals 3, and going to infinity. And you set your computer to sort of keep track of what the partial sums are as it does this. So you want it to produce a list of partial sums. And you know, maybe to get a good feel for this rate of divergence, what you want to do is let it run for a long time, collect a whole bunch of values. Say, you know, stop when it gets to some fairly large number.

So suppose we run this, and we ask it to stop when it gets to a partial sum that's greater than or equal to 5,000. You know, whatever. A moderately large integer or number. Doesn't have to be an integer. So it's going to stop as soon as it reaches a partial sum that's bigger than 5,000. So what I want to know, is how long is that going to take?

Now of course, that depends on your computer. So let's just say, you know, for sake of argument, that your computer-- this is generous to your computer. At least my computer doesn't go this fast-- that it takes 10 to the minus 20 seconds to add an additional term to the partial sums.

So suppose that your computer takes 10 to the minus 20 seconds for each additional summand. How long, how many terms of the series, how long are you going to wait around, waiting for this sum to get to 5,000?

So that's the question. Why don't you pause the video, take some time, try and work this out. This is probably not a quite good question for experimentation. I would advise against actually running it while you wait to continue with the video. But of course, you're welcome to try. But you can, you know, work this out, come back, and we'll talk about it together.

Welcome back. Hopefully you had some fun with this problem. Let's let's talk about it. So what we want to do, is we want to look at this series and say, you know, about how long it's going to take to get to 5,000. What we want-- which partial sum of this series is about equal to 5,000 for the first time? So let's give a notation there. So let's call S_N -- big N now, S big N is equal to N -th partial sum-- well, OK. Let me not make it the N -th partial sum. Let me just make it the sum when the top is big N . So going from 3 to big N of \log little n divided by n .

So first of all, the answer could be-- A priori, the answer could be forever. It might be that series converges to some number less than 5,000. So if that were to happen, then, you know, no matter how long you waited around,

you wouldn't get there. But we can check that that's not the case. So we can check that this series diverges.

So how can we check that? So the series diverges by the integral test, because the integral of $\log x$ over x dx from, say, 3 to infinity, though of course the bounds aren't-- the lower bound, 3, is not crucially important. This is something we know how to integrate. This is equal to $1/2 \log$ of x quantity squared, taken between, well, 3 and infinity. And anyway, you see that as x goes to infinity, \log of x goes to infinity, and so \log of x squared goes to infinity. So this diverges.

So the series does diverge to positive infinity. And as a result of its divergence to positive infinity, we know that eventually, we do get to 5,000. OK, so this is-- the answer is not an infinite amount of time. The answer is some finite amount of time. So now the question is, how much.

Now, the thing to remember about the integral test is not only can it tell you whether something converges or diverges, but it gives you a really good estimate of how quickly it converges or diverges.

So I have over here a little picture. This is the function y equals \log of x over x . So at x equals 1, this function is 0, and then it reaches a maximum, and then tapers down. And as x gets larger-- so it's a little hard to tell in this picture-- at x equals 6, this is still-- well, so the peak here is a little bit less than 0.4. At x equals 6, this, the function value is still bigger than a quarter. But it does eventually go down to 0, because x grows faster than $\log x$. And of course if you were interested, an interesting exercise might be really precise, you know, where, to figure out precisely, what is this peak, and what's the x -value, and what's the y -value of that peak point before it starts turning down. If you wanted something fun to work out.

And OK. But that doesn't actually have anything to do with our problem, which has to do with this integral test. So remember that the integral in question is the area under this curve from 3 onwards, and a partial sum is what you get when you take a bunch of these left endpoint rectangles. And because this function is decreasing, the rectangles completely cover the region. And so the integral test tells us that the integral from 3 to big N of our function, $\log x$ over x dx , is strictly less than-- well, we want the partial sum that goes up to, so whose right interval here ends at big N . So that's actually the N minus first partial sum. So maybe what I'll do, is I'll add a plus 1 here, and I'll make this the big N th partial sum is bigger than the sum from 3 to N plus 1 of the function.

And also, if you turn this around, and you took right endpoint rectangles instead, you see that if you chop off the first rectangle, then what you get is the-- you get an area less than the integral. So we also have an upper bound here. This is less than or equal to the very first term, which in our case is $\ln 3$ over 3 plus that same integral.

So this is this formula that Professor Jerison showed you in class that bounds the partial sums in between the integral. So here, where the integral, instead of being an improper integral, just goes up to N plus 1. So the

integral test gives us, it tells us that this diverges if and only if this does, but it also gives us explicit bounds for those two expressions, how they relate to each other. And so it says that the rate of divergence of S_N is very, very closely tied to the rate of divergence of this integral.

Well, so what? So OK. So we said, we know, you know, about how big this integral is. How big is it? Well, we saw that it's about $\log x$ squared between the bounds. So this is equal to, well, it's about, it's $1/2 \log$ of n plus 1 squared minus $1/2 \log 3$ squared is less than S_N , which is less than $\log 3$ over 3 plus the same thing. So this is $\log N$ plus 1 quantity squared minus $1/2 \log 3$ squared. And here this is $\log N$ plus 1 quantity squared. OK. So this right-hand side is equal to this right-hand side.

OK. So we have that S_N and this is a big N . So we have that S_N is about bounded by these-- sorry-- is definitely bounded by these two expressions. Another thing to note is that the constants here are pretty small. Right? This $1/2 \log 3$ squared and $\log 3$ over 3 times $1/2 \log 3$ squared, those are pretty small. From this point on in this problem, I'm going to stop worrying about precise constants like that, and I'm going to start using an approximate sign. So another way to write this is to say that S_N is approximately-- so this is like an equals sign, but both of them are curvy. So what I mean by this, it doesn't mean anything formally. So it's not a precise meaning. In this case, it's going to mean always about the same size. For purposes of this argument, it's going to be about the same. So this is about equal to $1/2 \ln$ of N plus 1 quantity squared. OK. And what we want--

So, OK. So, good. So we have a good approximation for about how big the N th partial sum is. Well, all right. So why? Well, because I know that what I want to do is I want to keep going until I reach a partial sum that has size about 5,000. So I want to know, how many terms do I need. Right? That's going to tell me how much time it requires, and I need to know how many terms there are. So I need to know, when S_N is about 5,000, how big is N ?

So coming back over here, so I need to take this equation, and I need to solve it for N in terms of S_N . Because I know that S_N is going to be about 5,000, so I want to know, how big is N going to be? So OK. So let's do that. So we can multiply by 2, take a square root, and exponentiate. And what we get-- so I'm going to move up here. So from this expression, what I get is that N plus 1 is approximately equal to-- so I'm going to multiply by 2, take a square root, and exponentiate, so that's e to the square root of $2 \cdot S_N$.

And you know what, I don't care about this plus 1. I'm going to forget about it. So N is approximately equal to e to the square root of $2 \cdot S_N$. Now in our case, we want to know particularly when S_N is equal to about 5,000. So when S_N is approximately equal to 5,000, this means that N is approximately-- well, 2 times 5,000 is 10,000, square root of 10,000 is 100, so N is approximately equal to e to the 100.

Now e is between 2 and 3-- so OK, so you know, I don't, I can't really give a very precise estimate of this off the

top of my head. But luckily I used a computer ahead of time to approximate it. So e to the 100 is about equal to 2 times 10 to the forty-third power. So that's the number, so that's about how big N is, order of magnitude.

So we need to do about 2 times 10 to the forty-third terms. So now that's the number of terms. Now, if you remember, back in our question, we were asked, on the assumption that each term takes 10 to the minus 20 seconds, how long is this going to take. So if each term requires 10 to the minus 20 seconds, this means we're dealing with 2 times 10 to the 23 seconds. That's how long you'll have to wait.

Now, this is a big number. Right? So let's think about how big a number this is. Because there are big numbers, and then there are ridiculously big numbers. So, a big number would be a number where like, maybe this 10 to the minus 20 is a bad estimate, right? I mean, maybe you are emperor of the world, and you can harness all the computing power in the entire world to work on this problem. And so then, in that case, maybe you can do, say, 10 to the 25 computations per second, something. Or whatever.

So how much of a dent would that make in this number? Like 2 times 10 to 23. OK. It's not going to be done before lunchtime, but how long will it take? So just to-- let's see. I have this written down here. OK. So for comparison, this is about equal to 50,000 times the age of the universe. So if you had, if you harnessed all the computing power on earth, you could probably get this done in a few billion years. So OK, so just a sense of just how slowly this thing is getting. If you wait, and you have all the computer power on earth adding terms of the series, in, say, a few billion years you will be able to figure out-- you will sum up terms that pass 5,000 in value. So OK. So I don't know. I find that amusing.

And remember that the key here to this whole problem was just that we were using this integral test here. And specifically, we were using it in the form that I have over here, this form that lets you bound the partial sums between two values of the integral. And because we have those very tight bounds, these approximations are all valid, and so we can really say, yes, this problem that I've described is really something you can't do in an afternoon or whatever.

So I'll stop there.